COT4420 Theory of Computation September 1, 2024

Problem Set 1
Due September 17, 2024

Each problem is worth 10 points. In the following problems, as needed, you can use black-box diagrams as discussed in class.

1. Assume that a *procedure* is formalized as a Turing machine, which in turn can be represented as a finite length string from a finite alphabet. Furthermore, *any* string over this alphabet is simply defined to be a Turing machine. Is the set of all Turing machines countable? If so, give a listing (1-1 map with the integers) of all Turing machines. Note, such a listing is called an "effective enumeration." If not explain why the set of all Turing machines is not countable.

2. Black-boxes diagrams were discussed in class to explain the notions of algorithms and procedures. For a box that represents a procedure, explain: (a) what is the input and where is chosen from, (b) what are the outputs and where are they chosen from, and (c) what does the box do. Do the same for an algorithm.

3. Show that if L_1 and L_2 are recursive languages, then the intersection of the two languages is also a recursive language.

4. Cantor's original result (theorem) is: for any nonempty set (whether finite or infinite), the cardinality of S is *strictly less* than that of its power set 2^S. Prove part of this by showing that there is a one-to-one (but not necessarily onto) map *f* from S to its power set. This would prove that $|S| \leq |2^S|$. To prove the theorem, as a second step, suppose we could also show the following: assume that there *is* a one-to-one and onto function *g* (from S to its powerset) and show that this assumption leads to a contradiction. You do not need to prove this second step. However, you must explain why this second step thus proves Cantor's theorem.

5. Prove that the set of all languages that are not recursively enumerable is not countable.

6. Consider the grammar $G = (\{S, A, B\}, \{a, b\}, P, S)$ with *P*:
   $S \to aAB \mid aBA \mid bAA \mid \lambda$
   $A \to aS \mid bAAA$
   $B \to aABB \mid aBAB \mid aBBA \mid bS$

   What is L(*G*)? Outline the reasons for your claim. What type of grammar is this?

7. Using induction, show that for every $x \in \{a,b\}^*$ such that *x* begins with *a* and ends with *b*, *x* contains the substring *ab*. Hint: use *string length* on which to do the induction.

8. Find grammars that generate the following languages:
   (a) $L = \{a^n b^m a^m b^n : n, m \geq 0\}$ where $V = \{a, b\}$.
   (b) $L = \{wcw^R : w \in \{a, b\}^+\}$, where $V = \{a, b, c\}$ (*note:* w^R *is the* reverse *of string w*).
   (c) $L = \{w : |w| \bmod 3 \geq |w| \bmod 2\}$ where $V = \{a\}$.