Lecture 7

Properties of regular languages

COT 4420

Theory of Computation

Section 4.1

Closure properties of regular languages

- If L_1 and L_2 are regular languages, then we prove that:

Union: $L_1 \cup L_2$

Concatenation: L_1L_2

Star: L_1^*

Reversal: L_1^R

Complement: $\overline{L_1}$

Intersection: $L_1 \cap L_2$

} Are regular Languages

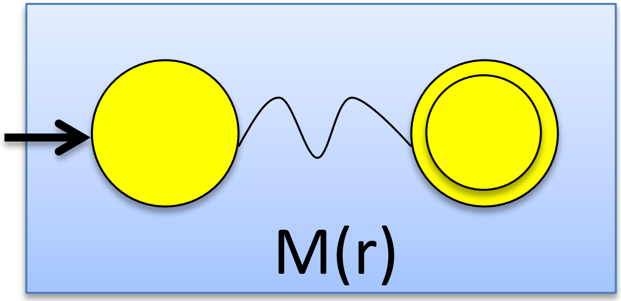
Closure properties of regular languages

- If L_1 and L_2 are regular languages, then we prove that under:

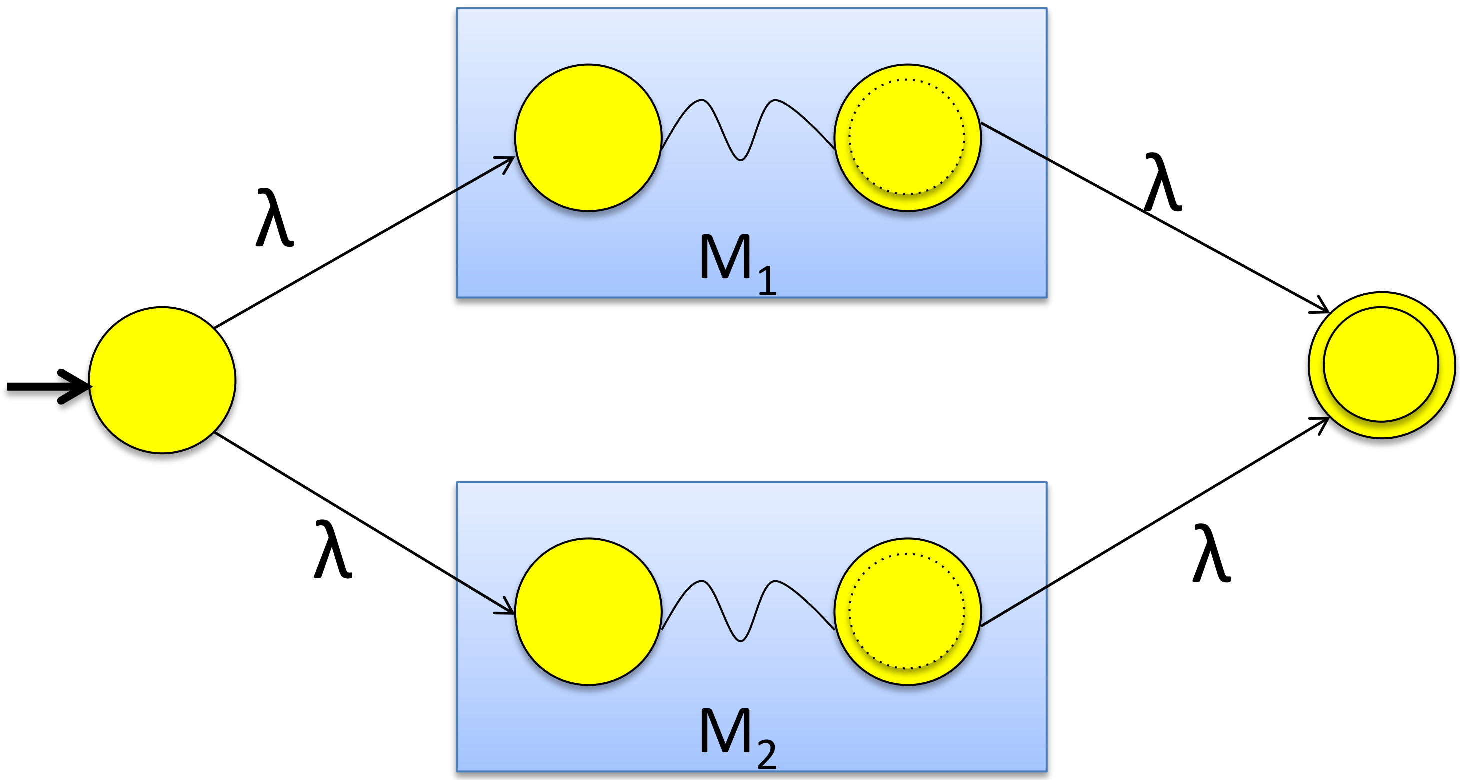
Substitution
Homomorphism
Inverse homomorphism
Right Quotient: L_1/L_2

} Are regular Languages

- Suppose this is the representation of an NFA accepting L_1.



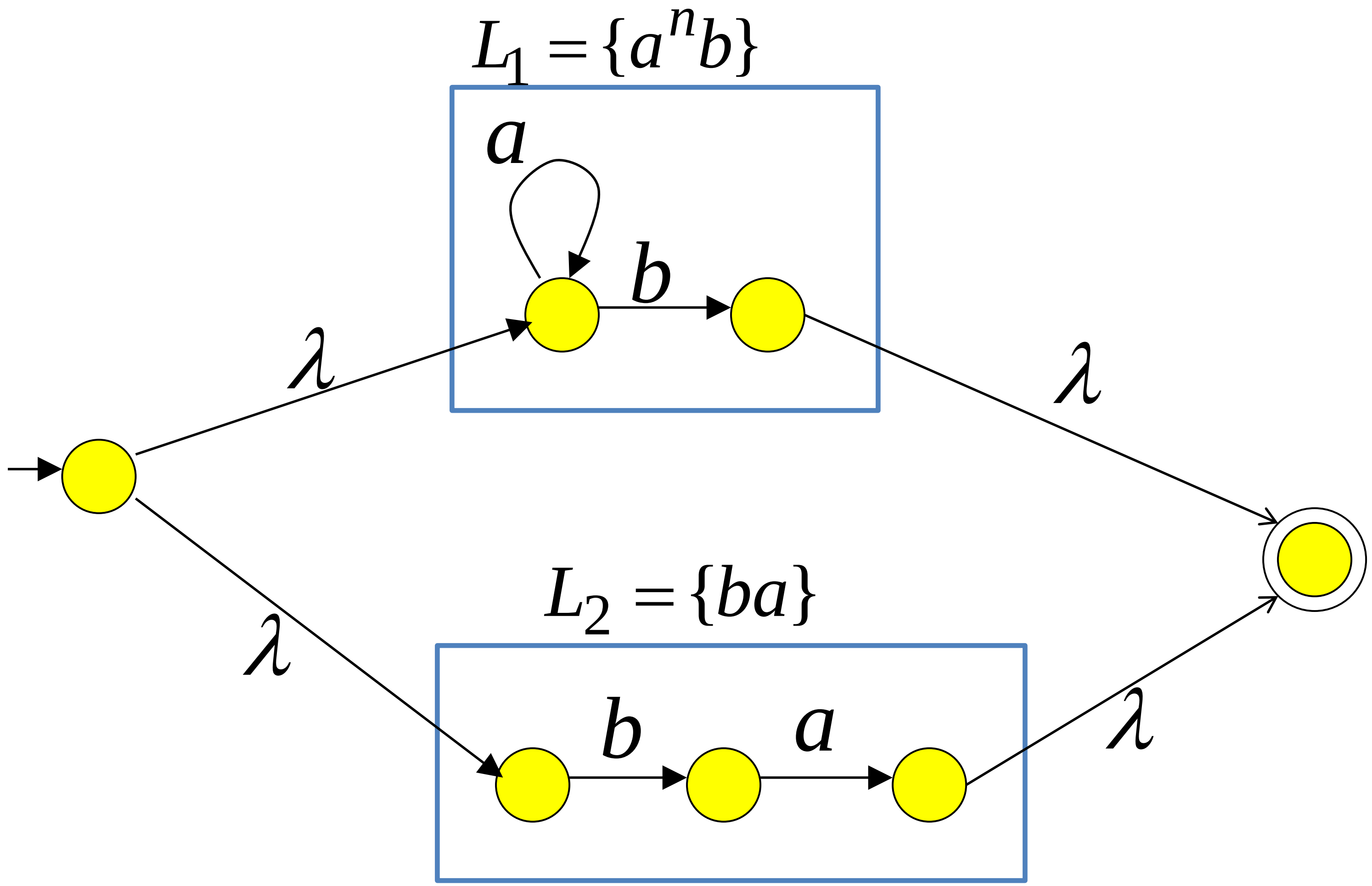
Union $L_1 \cup L_2$



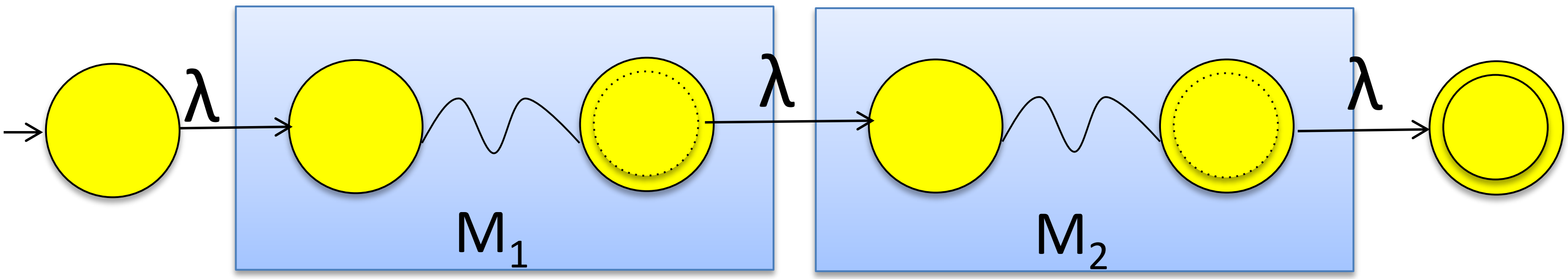
$$w \in L_1 \cup L_2 \iff w \in L_1 \text{ or } w \in L_2$$

Union - Example

$$L_1 \cup L_2 = \{a^n b\} \cup \{ba\}$$



Concatenation L_1L_2



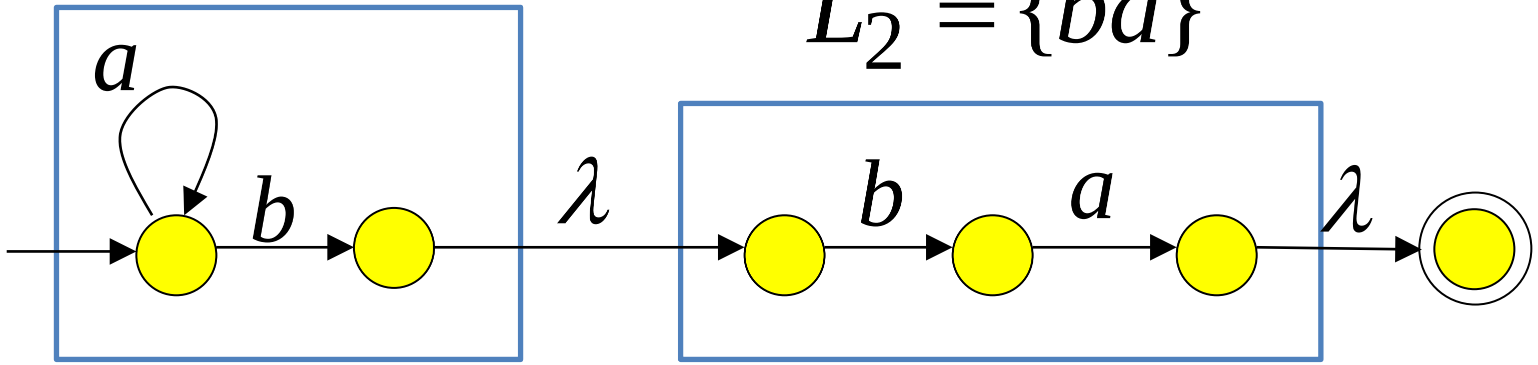
$$w \in L_1L_2 \Leftrightarrow w = w_1w_2 : \ w_1 \in L_1 \text{ and } w_2 \in L_2$$

Concatenation - Example

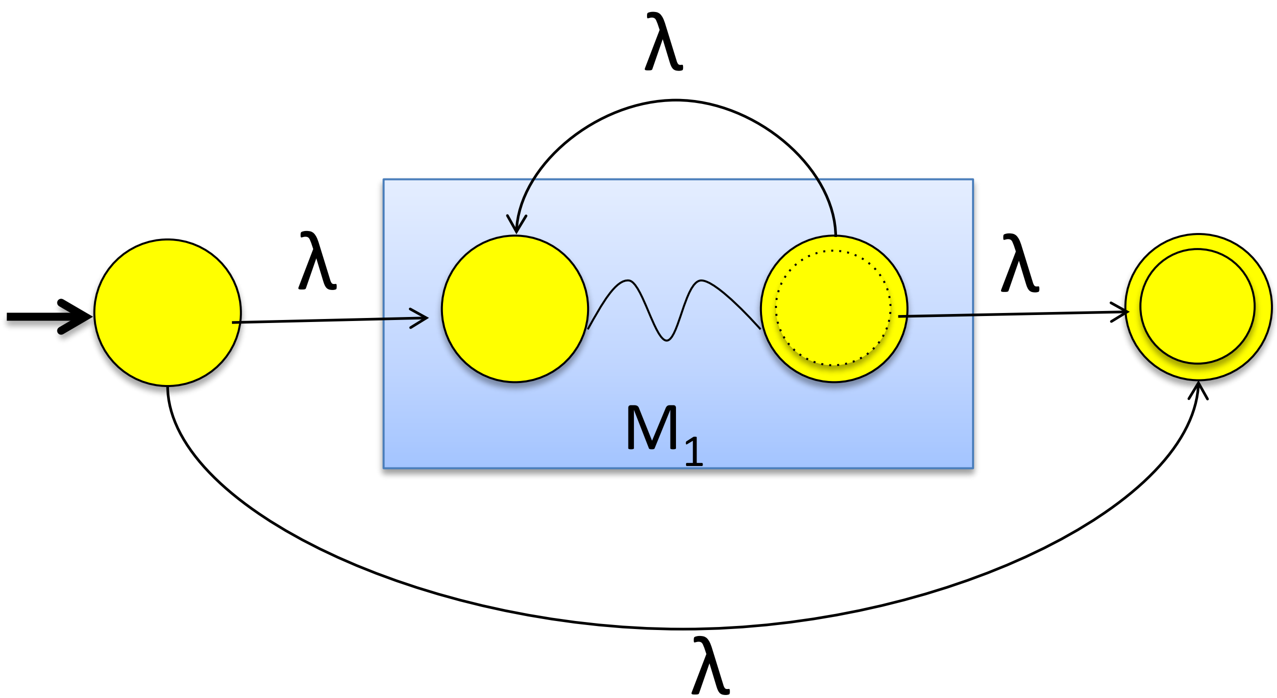
$$L_1 L_2 = \{a^n b\}\{ba\} = \{a^n bba\}$$

$$L_1 = \{a^n b\}$$

$$L_2 = \{ba\}$$



Star Operation L_1^*

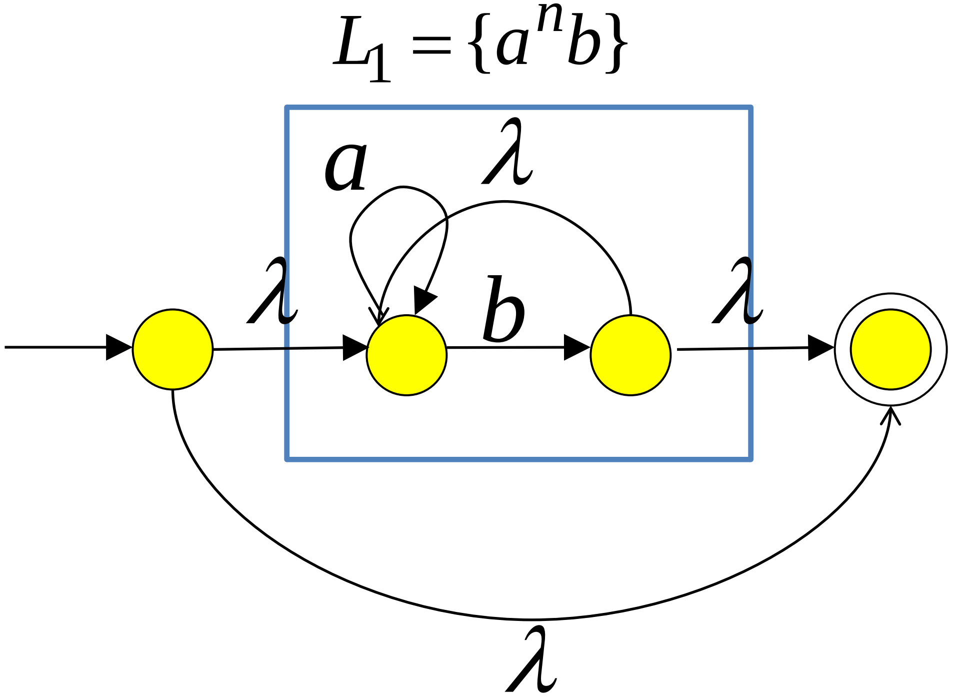


$$w \in L^* \iff w = w_1 w_2 \cdots w_k : \; w_i \in L$$

or $w = \lambda$

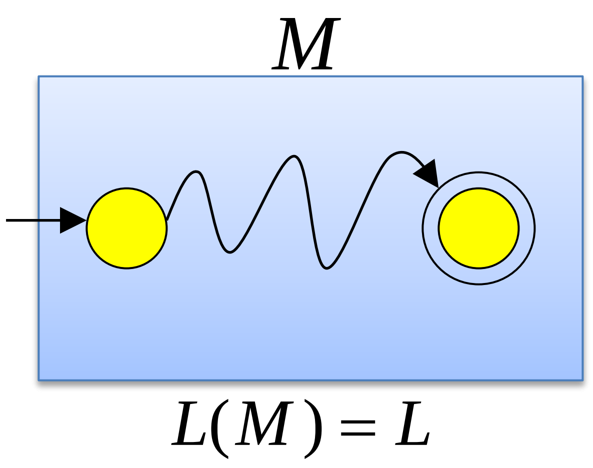
Star Operation - Example

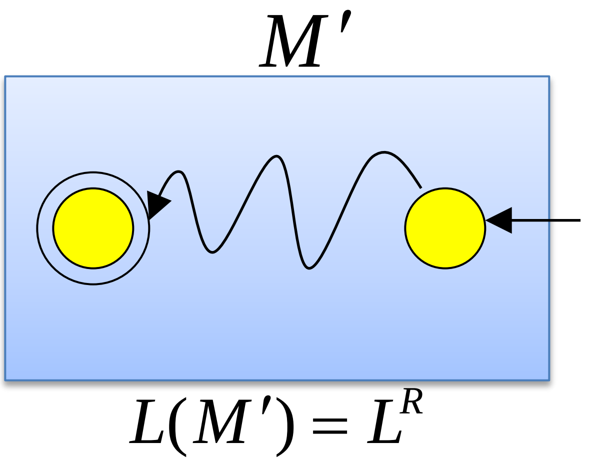
$$L_1^{\ *} = \{a^n b\}^*$$



Reverse L_1^R

- Make sure your NFA has single final state.
- Reverse all transitions
- Make the initial state accept state and make the accept state initial state.

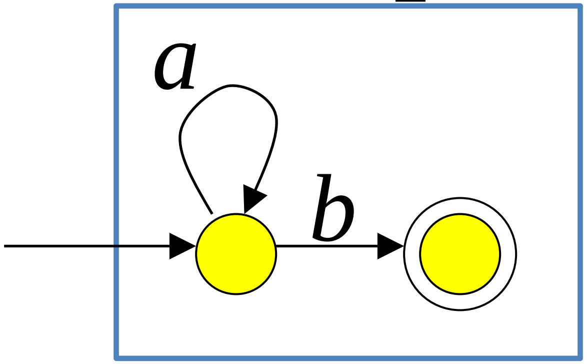




Reverse - Example

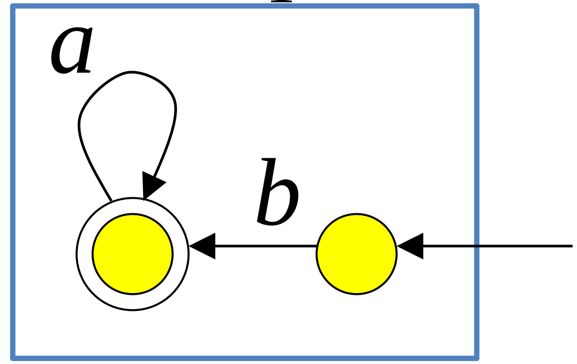
$$L_1 = \{a^n b\}$$

M_1



$$L_1^R = \{b a^n\}$$

M_1'

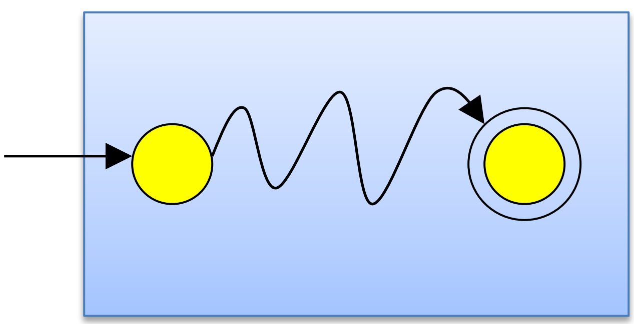


Complement $\overline{L}$

- Let M be the DFA that accepts L.

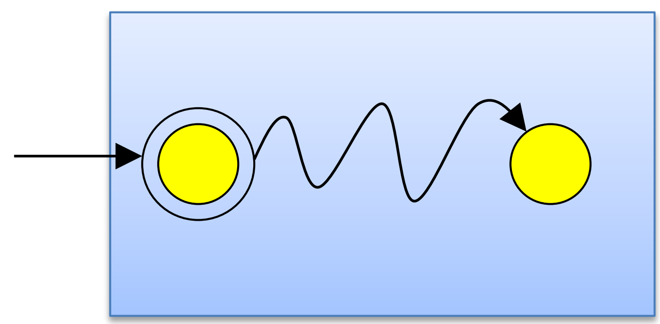
- Make final states non-final states and vice versa for M'

M



$L(M) = L$

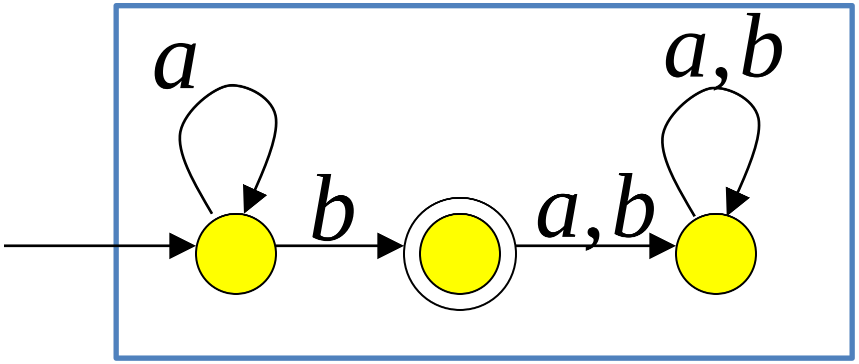
M'



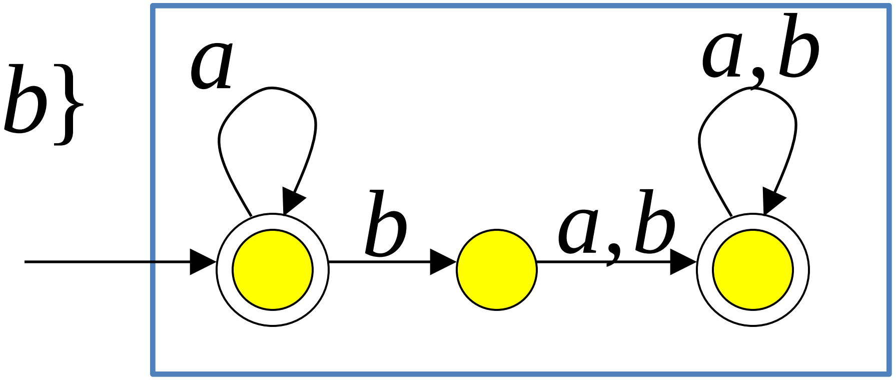
$L(M') = \overline{L}$

Complement - Example

$$L_1 = \{a^n b\}$$



$$\overline{L_1} = \{a,b\}^* - \{a^n b\}$$



Intersection $L_1 \cap L_2$

- De Morgan's Law: $L_1 \cap L_2 = \overline{\overline{L_1} \cup \overline{L_2}}$

L_1 and L_2 are regular ⇒ $\overline{L_1}$ and $\overline{L_2}$ are regular

⇒ $\overline{L_1} \cup \overline{L_2}$ is regular

⇒ $\overline{\overline{L_1} \cup \overline{L_2}}$ is regular

Substitution

- A substitution f is a mapping f: $\Sigma \rightarrow 2^{\Delta^*}$ (for some alphabet Δ). Thus f associates a language with each symbol of Σ.
- The mapping f is extended to strings as follows:

$$f(\lambda)=\lambda$$

$$f(xa) = f(x)f(a) \qquad x \in \Sigma^*, \quad a \in \Sigma$$

- The mapping f is also extended to languages by defining:

$$f(L) = \bigcup_{x \in L} f(x)$$

Substitution - Example

$\Sigma=\{0,1\}$ $\Delta=\{a,b\}$

Let $f(0) = ab^*$ and $f(1) = ac$

$f(011) = ab^*acac$

$f(011^*) = ab^*ac(ac)^*$

- If $f(a)$ is a regular language for $a \in \Sigma$, we call the substitution a regular substitution.

Closure under substitution

Theorem: Regular sets are closed under (regular) substitutions.

Let $R \subset \Sigma^*$ be a regular language. We need to show that $f(R)$ is a regular language.

For each $a \in \Sigma$, let $R_a \subset \Delta^*$ be a regular set such that $f(a) = R_a$. Select regular expressions denoting R and each R_a.

Replace each occurrence of **a** in the regular expression for R by the regular expression for R_a.

Closure under substitution

- The resulting regular expression is denoting $f(R)$. And it can be shown that:

$f(L_1 \cup L_2) = f(L_1) \cup f(L_2)$

$f(L_1 L_2) = f(L_1) f(L_2)$

$f(L_1^*) = (f(L_1))^*$

Homomorphism

- A homomorphism h is a substitution in which a single letter is replaced with a string.

  for a $\in \Sigma$, h(a) is a single string in Δ

  h: $\Sigma \rightarrow \Delta^*$

If $w = a_1a_2...a_n$ then $h(w) = h(a_1)h(a_2)...h(a_n)$

If L is a language on Σ, $h(L) = \{ h(w) : w \in L\}$ and is called its homomorphic image.

Homomorphism - Example

$\Sigma=\{0,1,2\}$ $\Delta=\{a,b\}$

$h(0) = ab$

$h(1) = b$

$h(2) = a$

Then $h(0110) = abbbab$

$h(122) = baa$

The homomorphic image of $L = \{0110,122\}$ is the language $h(L) = \{abbbab, baa\}$

Homomorphism

- Homomorphism is a substitution hence regular languages are closed under homomorphism.

- **Inverse homomorphism**:

Let $h: \Sigma \rightarrow \Delta^*$ be a homomorphism,

then $h^{-1}(w) = \{ x \mid h(x) = w\}$ for $w \in \Delta^*$.

$h^{-1}(L) = \{ x \mid h(x) \in L\}$ for $L \subset \Delta^*$

Inverse Homomorphism

Theorem: The class of regular sets is closed under inverse homomorphism.

- Let $h: \Sigma \rightarrow \Delta^*$ be a homomorphism and consider L a regular language in Δ^*. There must exists a dfa $M=(Q, \Delta, \delta, q_0, F)$ that accepts L. We construct $M_\Sigma = (Q, \Sigma, \delta', q_0, F)$ that accepts $h^{-1}(L)$ by defining $\delta'(q, a) = \delta(q, h(a))$ for $a \in \Sigma$.

By induction on $|x|$ we can show that $x \in L_\Sigma$ if and only if $h(x) \in L$.

Example

- Prove that L = { a^nba^n : $n \geq 1$} is not regular. Suppose we know hat $\{0^n1^n : n \geq 1\}$ is not regular.

$h_1(a) = a, \quad h_1(b) = ba \quad h_1(c)=a$

$h_2(a) = 0 \quad h_2(b) = 1 \quad h_2(c) = 1$

$h_1^{-1}(\{a^nba^n \mid n \geq 1\}) = (a+c)^nb(a+c)^{n-1}$

$h_1^{-1}(\{a^nba^n \mid n \geq 1\}) \cap a^*bc^* = \{a^nbc^{n-1}: n \geq 1\}$

$h_2(h_1^{-1}(\{a^nba^n \mid n \geq 1\}) \cap a^*bc^*) = \{0^n1^n: n \geq 1\}$

If { a^nba^n : $n \geq 1$} were regular since regular languages are closed under h and h^{-1} and intersection, $\{0^n1^n: n \geq 1\}$ must have been regular, a contradiction.

Right Quotient

- Let L_1 and L_2 be languages on the same alphabet. Then the right quotient of L_1 with L_2 is defined as:

$$L_1/L_2 = \{\, x: xy \in L_1 \text{ for some } y \in L_2\}$$

Example: $L_1 = 0^*10^*$ $L_2=10^*1$ $L_3=0^*1$

$L_1/L_3 = 0^*$

$L_2/L_3=10^*$

Right Quotient

Theorem: If L_1 and L_2 are regular languages, then L_1/L_2 is regular.

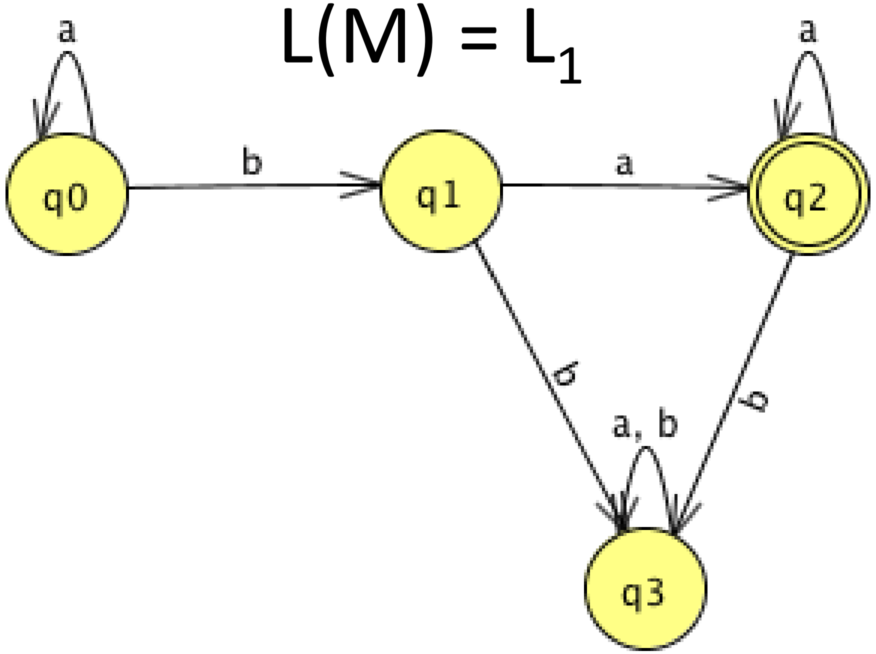
Suppose there is a dfa $M=(Q, \Sigma, \delta, q_0, F)$ such that $L_1 = L(M)$. We construct $M'=(Q, \Sigma, \delta, q_0, F')$ that accepts L_1/L_2.

For all states $q_i \in Q$ determine if there exists a $y \in L_2$ such that $\delta^*(q_i, y) = q_f \in F$. In that case we add q_i to F'.

Right Quotient - Example

$L_1 = L(a^*baa^*)$
$L_2 = L(ab^*)$

$L(M) = L_1$

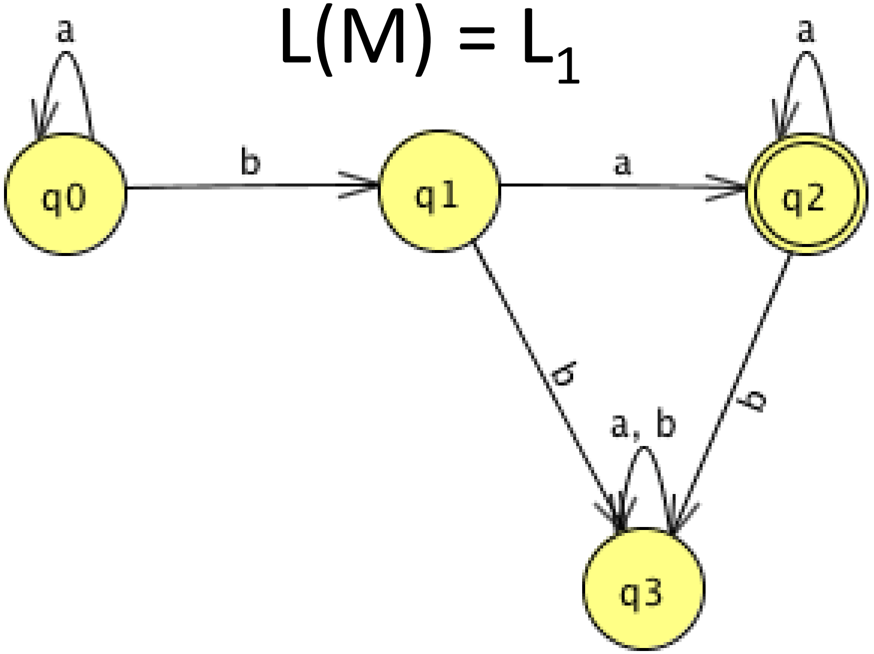


Right Quotient - Example

$L_1 = L(a^*baa^*)$
$L_2 = L(ab^*)$

For every state q_i determine if there is a $y \in L_2$ that $\delta^*(q_i, y) \in F$

$L(M) = L_1$



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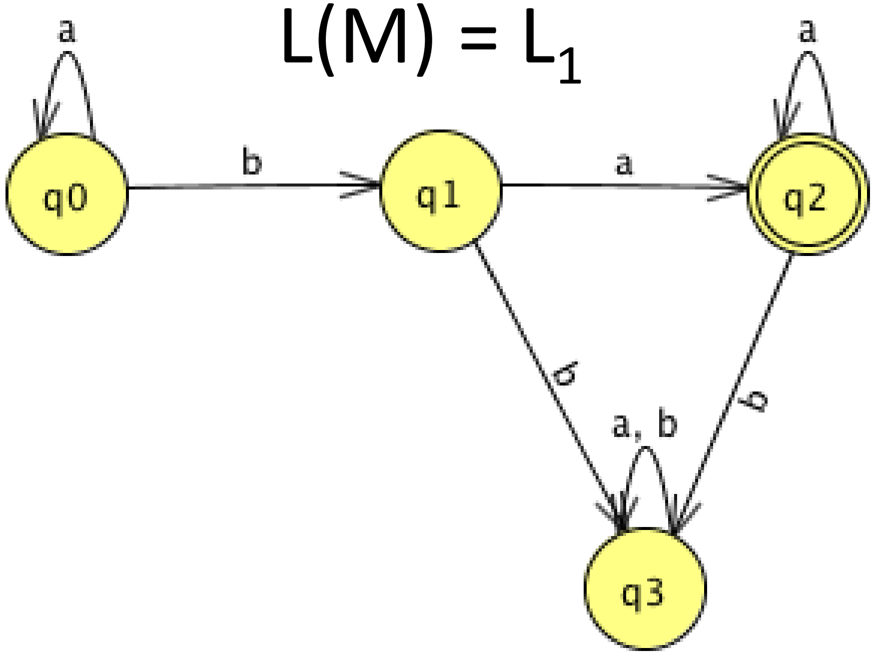

From q_0 ? No

Right Quotient - Example

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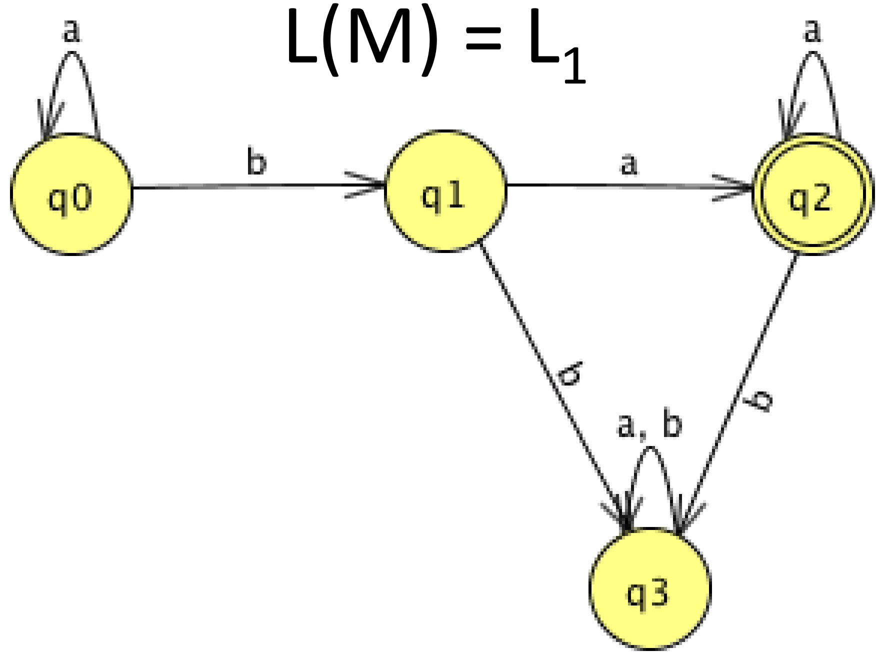

From q_0 ? No
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Right Quotient - Example

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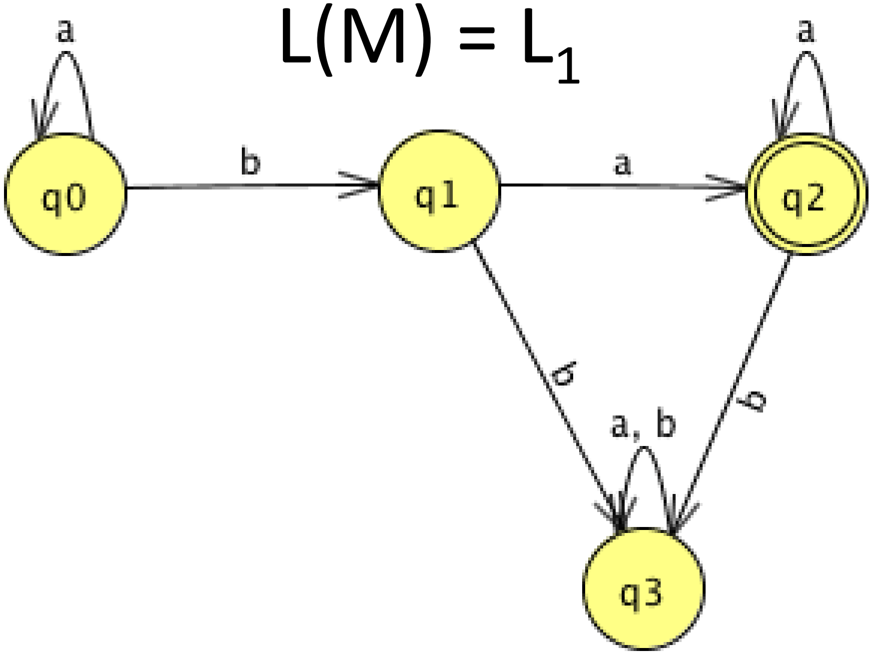

From q_0 ? No
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Right Quotient - Example

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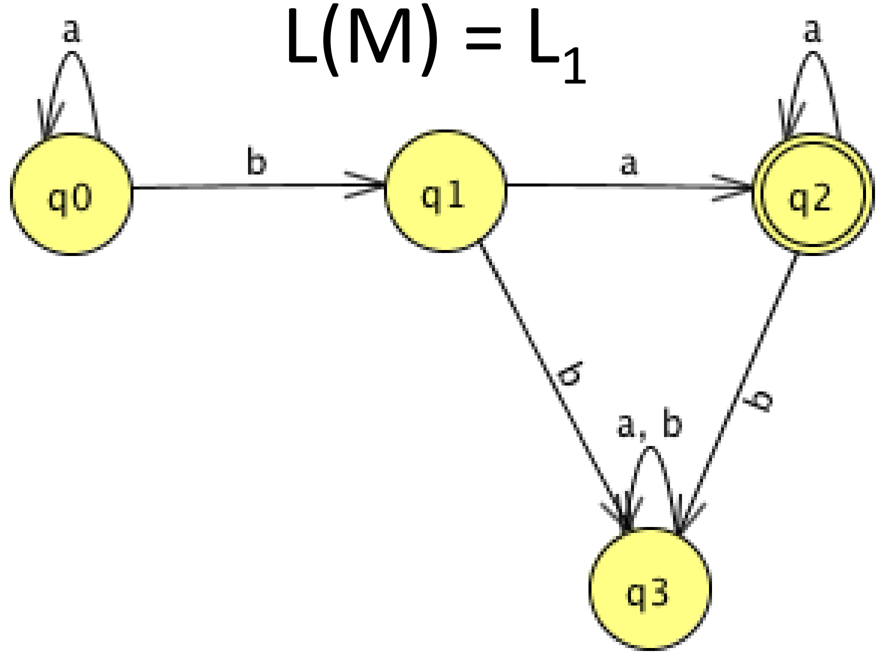
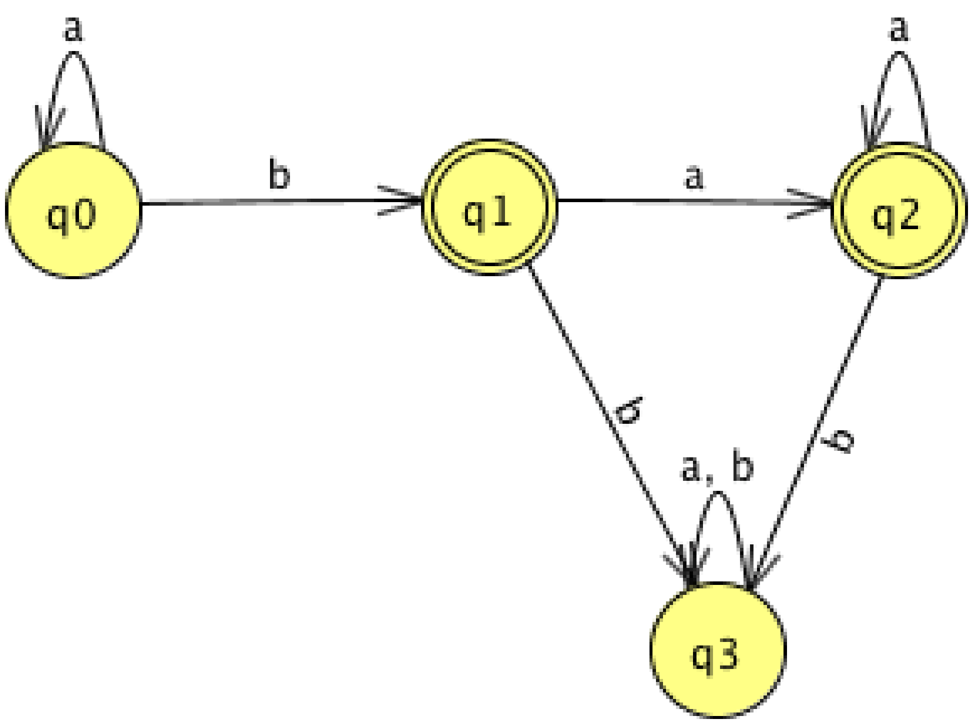


From q_0 ? No
From q_1 ? y = a
From q_2 ? y = a
From q_3 ? No

Right Quotient - Example



$$L_1/L_2 = L(a^*ba^*)$$