Lecture 10

Pumping Lemma Examples

COT 4420

Theory of Computation

Section 4.3

Pumping Lemma

Theorem: Let L be an infinite regular language. Then there exists some positive integer m such that any $w \in L$ with $|w| \geq m$ can be decomposed as $w = xyz$ with

$|xy| \leq m$ and

$|y| \geq 1$

Such that $w_i = xy^iz$ is also in L for all i = 0,1,2,...

Using pumping lemma to prove a language L is not regular

1. Assume the opposite : L is regular
2. The pumping lemma should hold for L
3. Use the pumping lemma to obtain a contradiction
4. Therefore, L is not regular

Using pumping lemma to prove a language L is not regular

1. Let m be the integer for pumping lemma

2. Pick a string $w \in L$, such that $|w| \geq m$

3. Decompose $w = xyz$ such that $|xy| \leq m$ & $|y| \geq 1$

4. Show that $w' = xy^i z$ is not in L for some i.

5. This results in contradiction since pumping lemma says $xy^i z \in L$ for all $i=0,1,2,3,\ldots$

Pumping Lemma
Example 2 $\{a^n b^{2n} : n \geq 0\}$

Question: Show that the language

$L = \{a^n b^{2n} : n \geq 0\}$ is not regular.

Answer: Proof by contradiction: assume L is regular. Since L is infinite we can apply pumping lemma!

Pumping Lemma
Example 2 $\{a^n b^{2n} : n \geq 0\}$

Let m be the integer for pumping lemma.

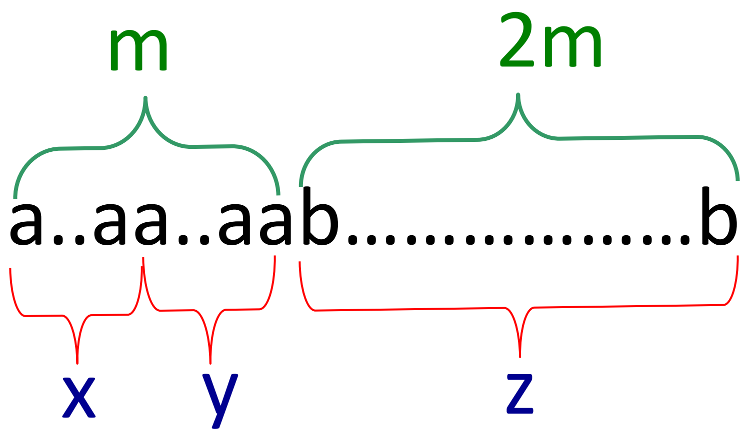
pick a string $w \in L$ such that $|w| \geq m$

$$w = a^m b^{2m}$$

Pumping Lemma
Example 2 $\{a^n b^{2n} : n \geq 0\}$

Decompose w = xyz such that $|xy| \leq m$ & $|y| \geq 1$

$W = a^m b^{2m} =$



$y = a^k, 1 \leq k \leq m$

Pumping Lemma
Example 2 $\{a^n b^{2n} : n \geq 0\}$

- Now we need to show that w' $= xy^iz$ is not in L for some i.

$m-k$ $2m$

$xy^0z =$ a..ab.................b

x z

$a^{m-k}b^{2m} \notin L$

CONTRADICTION!

Pumping Lemma
Example 2 $\{a^n b^{2n} : n \geq 0\}$

The assumption that L is a regular language is not true.

Therefore, L is not regular.

Pumping Lemma
Example 3 $\{ vv^R : v \in \Sigma^* \}$

Question: Show that the language

$L = \{ vv^R : v \in \Sigma^* \}$ is not regular.

Answer: Proof by contradiction: assume L is regular. Since L is infinite we can apply pumping lemma!

Pumping Lemma
Example 3 $\{ vv^R : v \in \Sigma^* \}$

Let m be the integer for pumping lemma.

pick a string $w \in L$ such that $|w| \geq m$

$$w = a^m b^m b^m a^m$$

Pumping Lemma
Example 3 $\{ vv^R : v \in \Sigma^* \}$

Decompose $w = xyz$ such that $|xy| \leq m$ & $|y| \geq 1$

m m m m

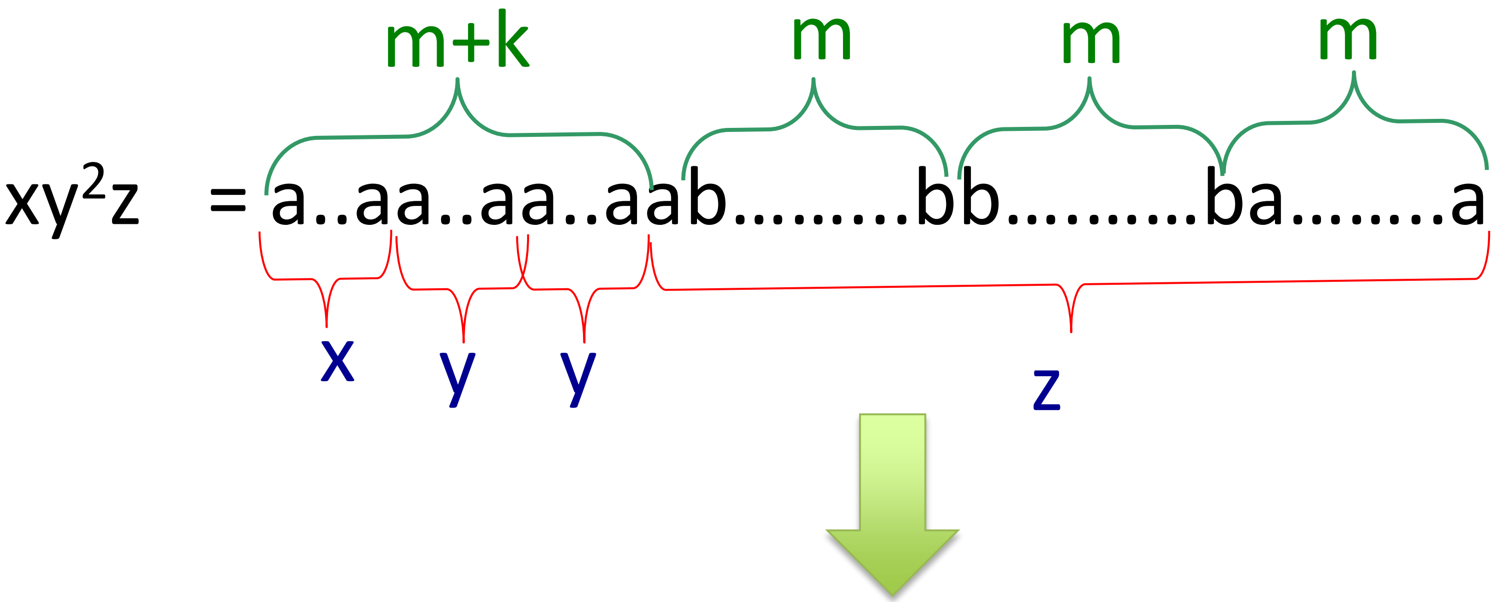
$W = a^m b^m b^m a^m$ = a..aa..aab........bb.........ba.......a

x y z

$y = a^k, 1 \leq k \leq m$

Pumping Lemma
Example 3 { $vv^R : v \in \Sigma^*$ }

- Now we need to show that w' = xy^iz is not in L for some i.



$a^{m+k}b^mb^ma^m \notin L$

CONTRADICTION!

Pumping Lemma
Example 3 { $vv^R : v \in \Sigma^*$}

The assumption that L is a regular language is not true.

Therefore, L is not regular.

Pumping Lemma
Example 4 $\{a^{n!} : n \geq 0\}$

Question: Show that the language $L = \{a^{n!} : n \geq 0\}$ is not regular.

Answer: Proof by contradiction: assume L is regular. Since L is infinite we can apply pumping lemma!

Pumping Lemma
Example 4 $\{a^{n!} : n \geq 0\}$

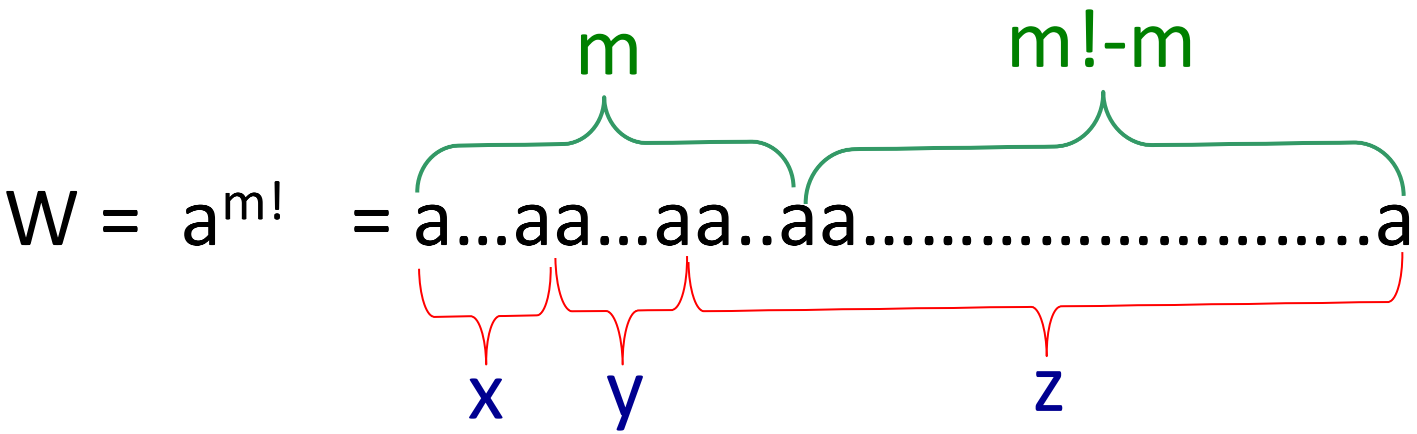
Let m be the integer for pumping lemma.

pick a string $w \in L$ such that $|w| \geq m$

$$w = a^{m!}$$

Pumping Lemma
Example 4 $\{a^{n!} : n \geq 0\}$

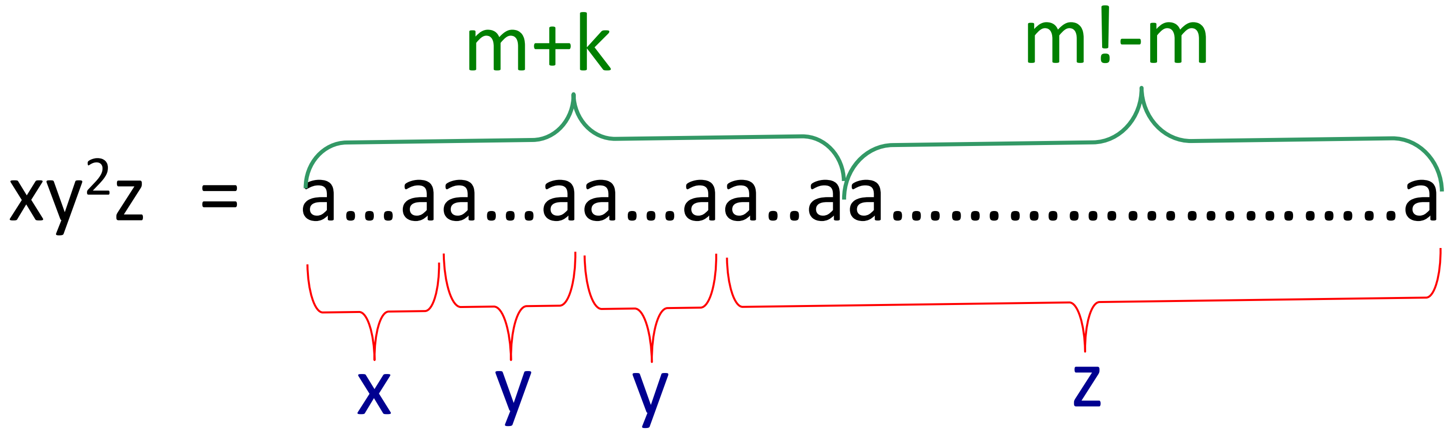
Decompose $w = xyz$ such that $|xy| \leq m$ & $|y| \geq 1$



$y = a^k$, $1 \leq k \leq m$

Pumping Lemma
Example 4 $\{a^{n!} : n \geq 0\}$

- Now we need to show that w' $= xy^iz$ is not in L for some i.



$$a^{m!+k} \overset{?}{\in} L$$

Pumping Lemma
Example 4 $\{a^{n!} : n \geq 0\}$

- In order for $a^{m!+k}$ to be in L, there must exists p such that $m!+k = p!$

However: $1 \leq k \leq m$, for $m > 1$

$$
\begin{aligned}
m!+k &\leq m!+m \\
&\leq m! + m! \\
&< m!m + m! \\
&= m!(m+1) \\
&= (m+1)!
\end{aligned}
$$

$m!+k < (m+1)!$

⇓

$m!+k \neq p!$ for any p

Pumping Lemma
Example 4 $\{a^{n!} : n \geq 0\}$

$a^{m!+k} \notin L$ CONTRADICTION!

The assumption that L is a regular language is not true.

Therefore, L is not regular.

Pumping Lemma
Example 5

Question: Show that the language L = { a^{i^2}: i ≥ 1} is not regular.

Answer: Proof by contradiction: assume L is regular. Since L is infinite we can apply pumping lemma!

Pumping Lemma
Example 5

Let m be the integer for pumping lemma.

pick a string $w \in L$ such that $|w| \geq m$

$$w = a^{m^2}$$

Pumping Lemma
Example 5

Decompose w = xyz such that $|xy| \leq m$ & $|y| \geq 1$

$$W = a^{m^2} = \overbrace{\underbrace{a\ldots a}_{x}\underbrace{a\ldots a}_{y}a..a}^{m}\underbrace{\ldots\ldots\ldots\ldots\ldots\ldots a}_{z}$$

$y = a^k$, $1 \leq k \leq m$

Pumping Lemma
Example 5

- Now we need to show that w' = xy^iz is not in L for some i.

m+k

xy^2z = a...aa...aa...aa..aa.......................a

x y y z

$$a^{m^2+k} \overset{?}{\in} L$$

Pumping Lemma
Example 5

$1 \leq k \leq m \quad , \quad \text{for } m > 1$

$$
\begin{aligned}
m^2 + k &\leq m^2 + m \\
&= m(m+1) \\
&< (m+1)^2
\end{aligned}
$$

$m^2 + k < (m+1)^2$

$m^2 + k \neq p^2 \quad \text{for any } p$