

Lecture 10

Pumping Lemma Examples

COT 4420

Theory of Computation

Pumping Lemma

Theorem: Let L be an infinite regular language. Then there exists some positive integer m such that any $w \in L$ with $|w| \geq m$ can be decomposed as $w = xyz$ with

$$|xy| \leq m \quad \text{and}$$

$$|y| \geq 1$$

Such that $w_i = xy^iz$ is also in L for all $i = 0, 1, 2, \dots$

Using pumping lemma to prove a language L is not regular

1. Assume the opposite : L is regular
2. The pumping lemma should hold for L
3. Use the pumping lemma to obtain a contradiction
4. Therefore, L is not regular

Using pumping lemma to prove a language L is not regular

1. Let m be the integer for pumping lemma
2. Pick a string $w \in L$, such that $|w| \geq m$
3. Decompose $w = xyz$ such that $|xy| \leq m$ & $|y| \geq 1$
4. Show that $w' = xy^iz$ is not in L for some i .
5. This results in contradiction since pumping lemma says $xy^iz \in L$ for all $i=0,1,2,3,\dots$

Pumping Lemma

Example 2 $\{a^n b^{2n} : n \geq 0\}$

Question: Show that the language

$L = \{a^n b^{2n} : n \geq 0\}$ is not regular.

Answer: Proof by **contradiction**: assume L is regular. Since L is infinite we can apply pumping lemma!

Pumping Lemma

Example 2 $\{a^n b^{2n} : n \geq 0\}$

Let m be the integer for pumping lemma.

pick a string $w \in L$ such that $|w| \geq m$

$$w = a^m b^{2m}$$

Pumping Lemma

Example 2 $\{a^n b^{2n} : n \geq 0\}$

Decompose $w = xyz$ such that $|xy| \leq m$ & $|y| \geq 1$

$$W = a^m b^{2m} = \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{2m} b$$

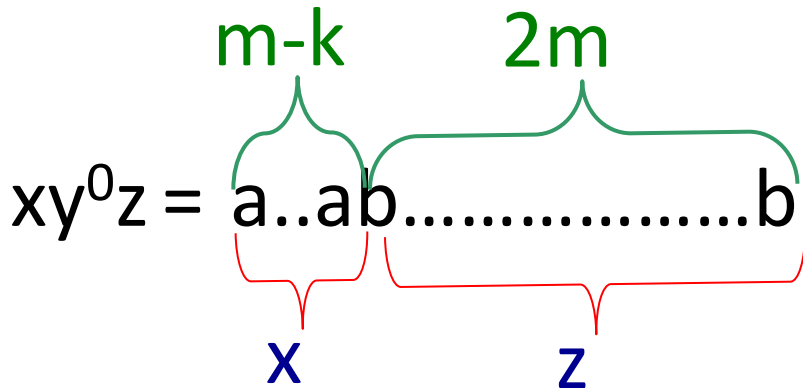
x y z

$$y = a^k, 1 \leq k \leq m$$

Pumping Lemma

Example 2 $\{a^n b^{2n} : n \geq 0\}$

- Now we need to show that $w' = xy^i z$ is not in L for some i .



$$a^{m-k} b^{2m} \notin L$$

CONTRADICTION!

Pumping Lemma

Example 2 $\{a^n b^{2n} : n \geq 0\}$

The assumption that L is a regular language is not true.

Therefore, L is not regular.

Pumping Lemma

Example 3 $\{ vv^R : v \in \Sigma^* \}$

Question: Show that the language

$L = \{ vv^R : v \in \Sigma^* \}$ is not regular.

Answer: Proof by **contradiction**: assume L is regular. Since L is infinite we can apply pumping lemma!

Pumping Lemma

Example 3 $\{ vv^R : v \in \Sigma^* \}$

Let m be the integer for pumping lemma.

pick a string $w \in L$ such that $|w| \geq m$

$$w = a^m b^m b^m a^m$$

Pumping Lemma

Example 3 $\{ vv^R : v \in \Sigma^* \}$

Decompose $w = xyz$ such that $|xy| \leq m$ & $|y| \geq 1$

$$W = a^m b^m b^m a^m = \underbrace{a \dots a}_{m} \underbrace{a \dots a}_{m} \underbrace{b \dots b}_{m} \underbrace{b \dots b}_{m} \underbrace{a \dots a}_{m}$$

x y z

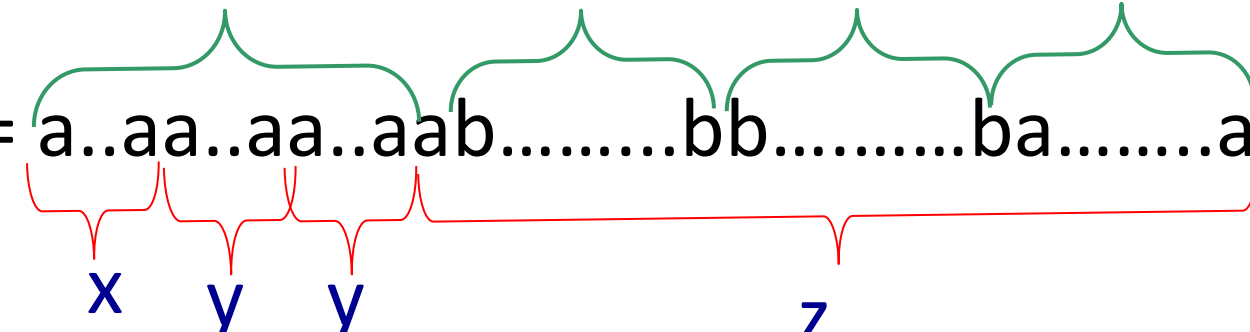
$$y = a^k, 1 \leq k \leq m$$

Pumping Lemma

Example 3 $\{ vv^R : v \in \Sigma^* \}$

- Now we need to show that $w' = xy^iz$ is not in L for some i .

$$xy^2z = \underbrace{a \dots a}_{m+k} \underbrace{a \dots a}_{m} \underbrace{b \dots b}_{m} \underbrace{b \dots b}_{m} \underbrace{a \dots a}_{m}$$





$$a^{m+k} b^m b^m a^m \notin L$$

CONTRADICTION!

Pumping Lemma

Example 3 $\{ vv^R : v \in \Sigma^* \}$

The assumption that L is a regular language is not true.

Therefore, L is not regular.

Pumping Lemma

Example 4 $\{a^{n!} : n \geq 0\}$

Question: Show that the language $L = \{a^{n!} : n \geq 0\}$ is not regular.

Answer: Proof by **contradiction**: assume L is regular. Since L is infinite we can apply pumping lemma!

Pumping Lemma

Example 4 $\{a^{n!} : n \geq 0\}$

Let m be the integer for pumping lemma.

pick a string $w \in L$ such that $|w| \geq m$

$$w = a^{m!}$$

Pumping Lemma

Example 4 $\{a^{n!} : n \geq 0\}$

Decompose $w = xyz$ such that $|xy| \leq m$ & $|y| \geq 1$

$$W = a^{m!} = \underbrace{a \dots a}_{m} \dots \underbrace{a \dots a}_{m! - m} \dots a$$

The diagram shows the string $a \dots a$ with a green bracket above the first m characters and another green bracket above the next $m! - m$ characters. Below the string, a red bracket groups the first m characters into x , the next k characters into y , and the remaining characters into z .

$$y = a^k, 1 \leq k \leq m$$

Pumping Lemma

Example 4 $\{a^{n!} : n \geq 0\}$

- Now we need to show that $w' = xy^iz$ is not in L for some i .

$$xy^2z = \underbrace{a \dots a}_{x} \underbrace{a \dots a}_{y} \underbrace{a \dots a}_{y} \underbrace{a \dots a}_{z}$$

$m+k$ $m!-m$

$$a^{m!+k} \stackrel{?}{\in} L$$

Pumping Lemma

Example 4 $\{a^{n!} : n \geq 0\}$

- In order for $a^{m!+k}$ to be in L , there must exist p such that $m!+k = p!$

However: $1 \leq k \leq m$, for $m > 1$

$$\begin{aligned} m!+k &\leq m!+m \\ &\leq m! + m! \\ &< m!m + m! \\ &= m!(m+1) \\ &= (m+1)! \end{aligned}$$

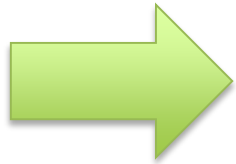
$$m!+k < (m+1)!$$



$$m!+k \neq p! \quad \text{for any } p$$

Pumping Lemma

Example 4 $\{a^{n!} : n \geq 0\}$



$a^{m!+k} \notin L$ **CONTRADICTION!**

The assumption that L is a regular language is not true.

Therefore, L is not regular.

Pumping Lemma

Example 5

Question: Show that the language $L = \{a^{i^2} : i \geq 1\}$ is not regular.

Answer: Proof by **contradiction**: assume L is regular. Since L is infinite we can apply pumping lemma!

Pumping Lemma

Example 5

Let m be the integer for pumping lemma.

pick a string $w \in L$ such that $|w| \geq m$

$$w = a^{m^2}$$

Pumping Lemma

Example 5

Decompose $w = xyz$ such that $|xy| \leq m$ & $|y| \geq 1$

$$W = a^{m^2} = \underbrace{a \dots a a \dots a a \dots a a \dots a \dots a}_{m} \dots a$$

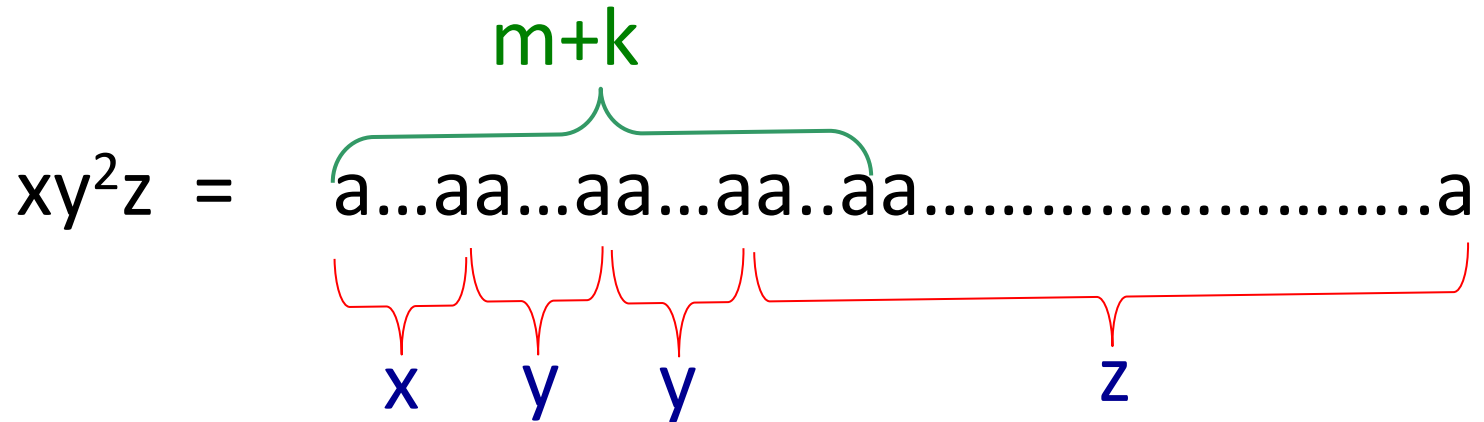
The diagram illustrates the decomposition of the string $w = a^{m^2}$ into xyz . A green bracket above the string indicates that the first m characters are grouped together. A red bracket below the string indicates that the string is partitioned into three parts: x (the first m characters), y (the next k characters), and z (the remaining characters).

$$y = a^k, 1 \leq k \leq m$$

Pumping Lemma

Example 5

- Now we need to show that $w' = xy^iz$ is not in L for some i .



$$a^{m^2+k} \stackrel{?}{\in} L$$

Pumping Lemma

Example 5

$$1 \leq k \leq m, \text{ for } m > 1$$

$$\begin{aligned} m^2 + k &\leq m^2 + m \\ &= m(m+1) \\ &< (m+1)^2 \end{aligned}$$

$$m^2 + k < (m+1)^2$$



$$m^2 + k \neq p^2 \text{ for any } p$$