Introduction to the Mathematics of Regression Part 1

by

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for

Introduction to Artificial Intelligence
(CAP 4601)

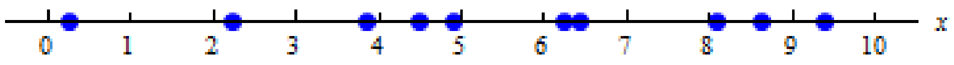
Agenda

- Sum of Parabolas
- Minima: Weighted Sample Mean
- Sample Mean
- Sum of Squared Differences, Biased Sample Variance, and Unbiased Sample Variance
- Biased and Unbiased Sample Standard Deviation
- Sum of 2D Parabolas
- 2D Sample Mean
- Sum of the Product of Differences, Biased Sample Covariance, and Unbiased Sample Covariance
- Line
- Linear Regression
- Minima: Sum of Squared Differences and Sum of the Product of Differences
- Independent Variable Bias
- Online Mean
- One Way To Program Simple Linear Regression

Data

- Given:

```
{
    0.255193,
    2.23273,
    3.85555,
    4.49383,
    4.90807,
    6.24521,
    6.41867,
    8.09172,
    8.61902,
    9.39342
}
```



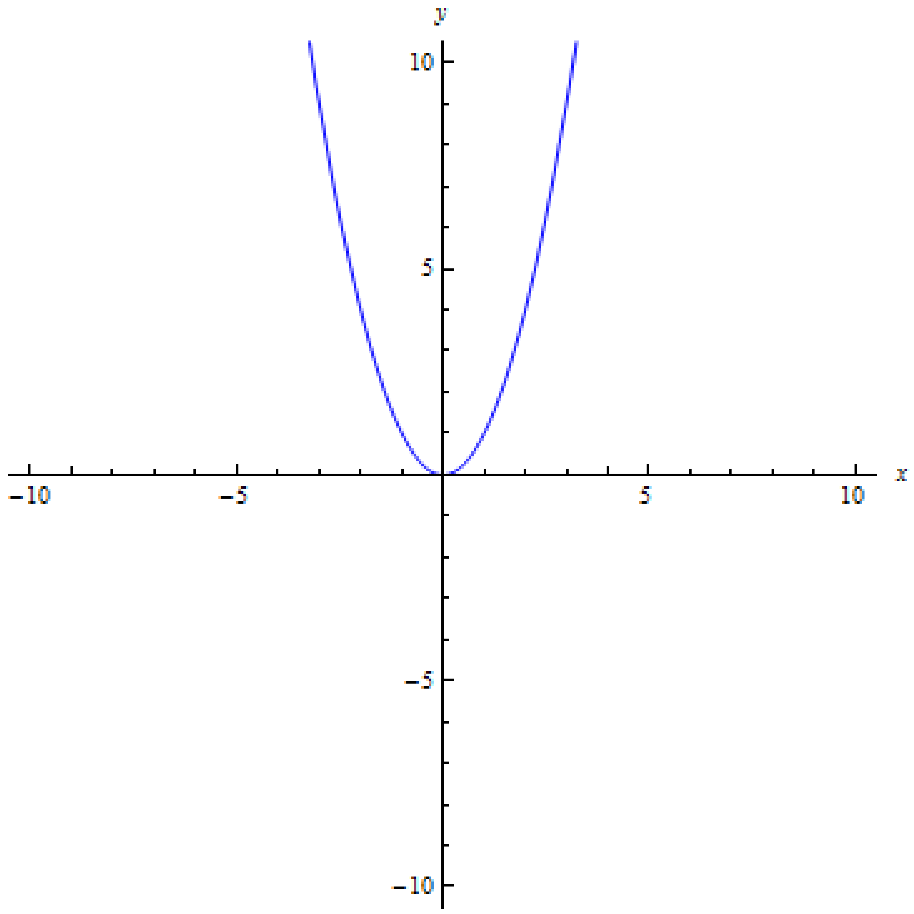
Parabola

$$y = w\left(x - x_0\right)^2 + y_0$$

Let $w = 1$, $x_0 = 0$, and $y_0 = 0$.

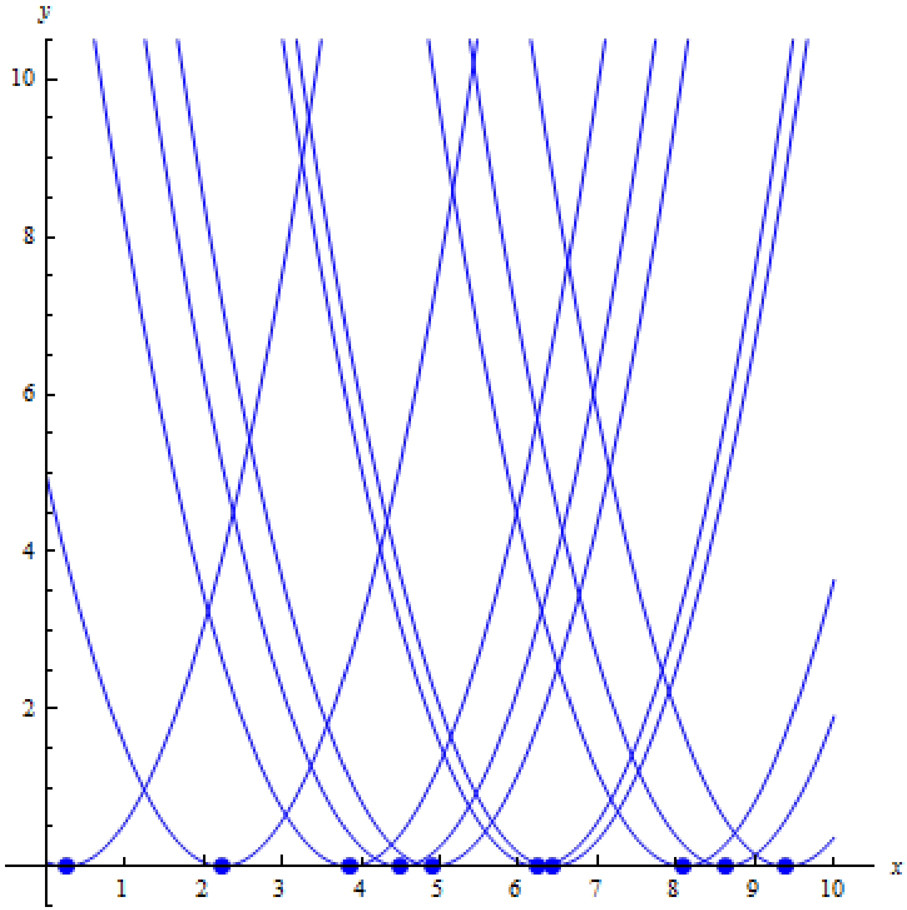
$$y = 1 \cdot \left(x - 0\right)^2 + 0$$

$$y = x^2$$



Parabolas

- For each datum, place a parabola centered on that value.



Sum of Parabolas

$$y = w\left(x - x_0\right)^2 + y_0 \quad \text{and} \quad y_0 = 0$$

$$y = \sum_{i=1}^{n} w_i\left(x - x_i\right)^2$$

Sum of Parabolas

$$w_1\left(x-x_1\right)^2+w_2\left(x-x_2\right)^2+w_3\left(x-x_3\right)^2+w_4\left(x-x_4\right)^2+w_5\left(x-x_5\right)^2+$$
$$w_6\left(x-x_6\right)^2+w_7\left(x-x_7\right)^2+w_8\left(x-x_8\right)^2+w_9\left(x-x_9\right)^2+w_{10}\left(x-x_{10}\right)^2$$

$$w_1x^2-2w_1x_1x+w_1x_1^2+w_2x^2-2w_2x_2x+w_2x_2^2+w_3x^2-2w_3x_3x+w_3x_3^2+w_4x^2-2w_4x_4x+w_4x_4^2+$$
$$w_5x^2-2w_5x_5x+w_5x_5^2+w_6x^2-2w_6x_6x+w_6x_6^2+w_7x^2-2w_7x_7x+w_7x_7^2+w_8x^2-2w_8x_8x+w_8x_8^2+$$
$$w_9x^2-2w_9x_9x+w_9x_9^2+w_{10}x^2-2w_{10}x_{10}x+w_{10}x_{10}^2$$

$$\left(w_1+w_2+w_3+w_4+w_5+w_6+w_7+w_8+w_9+w_{10}\right)x^2+$$
$$\left(-2w_1x_1-2w_2x_2-2w_3x_3-2w_4x_4-2w_5x_5-2w_6x_6-2w_7x_7-2w_8x_8-2w_9x_9-2w_{10}x_{10}\right)x+$$
$$\left(w_1x_1^2+w_2x_2^2+w_3x_3^2+w_4x_4^2+w_5x_5^2+w_6x_6^2+w_7x_7^2+w_8x_8^2+w_9x_9^2+w_{10}x_{10}^2\right)$$

Minima: Weighted Sample Mean

$$\frac{d}{dx}\left[\begin{array}{c}\left(w_1+w_2+w_3+w_4+w_5+w_6+w_7+w_8+w_9+w_{10}\right)x^2+\\ \left(-2w_1x_1-2w_2x_2-2w_3x_3-2w_4x_4-2w_5x_5-2w_6x_6-2w_7x_7-2w_8x_8-2w_9x_9-2w_{10}x_{10}\right)x+\\ \left(w_1x_1^2+w_2x_2^2+w_3x_3^2+w_4x_4^2+w_5x_5^2+w_6x_6^2+w_7x_7^2+w_8x_8^2+w_9x_9^2+w_{10}x_{10}^2\right)\end{array}\right]=0$$

$$\begin{array}{c}2\left(w_1+w_2+w_3+w_4+w_5+w_6+w_7+w_8+w_9+w_{10}\right)x+\\ \left(-2w_1x_1-2w_2x_2-2w_3x_3-2w_4x_4-2w_5x_5-2w_6x_6-2w_7x_7-2w_8x_8-2w_9x_9-2w_{10}x_{10}\right)\end{array}=0$$

$$x=\frac{-\left(-2w_1x_1-2w_2x_2-2w_3x_3-2w_4x_4-2w_5x_5-2w_6x_6-2w_7x_7-2w_8x_8-2w_9x_9-2w_{10}x_{10}\right)}{2\left(w_1+w_2+w_3+w_4+w_5+w_6+w_7+w_8+w_9+w_{10}\right)}$$

$$x=\frac{w_1x_1+w_2x_2+w_3x_3+w_4x_4+w_5x_5+w_6x_6+w_7x_7+w_8x_8+w_9x_9+w_{10}x_{10}}{w_1+w_2+w_3+w_4+w_5+w_6+w_7+w_8+w_9+w_{10}}$$

Minima: Weighted Sample Mean

$$\frac{d^2}{dx^2}\left[\begin{array}{c}\left(w_1+w_2+w_3+w_4+w_5+w_6+w_7+w_8+w_9+w_{10}\right)x^2+\\ \left(-2w_1x_1-2w_2x_2-2w_3x_3-2w_4x_4-2w_5x_5-2w_6x_6-2w_7x_7-2w_8x_8-2w_9x_9-2w_{10}x_{10}\right)x+\\ \left(w_1x_1^2+w_2x_2^2+w_3x_3^2+w_4x_4^2+w_5x_5^2+w_6x_6^2+w_7x_7^2+w_8x_8^2+w_9x_9^2+w_{10}x_{10}^2\right)\end{array}\right]$$

$$2\left(w_1+w_2+w_3+w_4+w_5+w_6+w_7+w_8+w_9+w_{10}\right)$$

$$\text{Since } 0<w_i \text{ for } i=1,2,\ldots,10$$

$$0<2\left(w_1+w_2+w_3+w_4+w_5+w_6+w_7+w_8+w_9+w_{10}\right)$$

Sample Mean

Let $w_i = 1$ for $i = 1, 2, \ldots, 10$.

$$x = \frac{w_1x_1 + w_2x_2 + w_3x_3 + w_4x_4 + w_5x_5 + w_6x_6 + w_7x_7 + w_8x_8 + w_9x_9 + w_{10}x_{10}}{w_1 + w_2 + w_3 + w_4 + w_5 + w_6 + w_7 + w_8 + w_9 + w_{10}}$$

$$x = \frac{1x_1 + 1x_2 + 1x_3 + 1x_4 + 1x_5 + 1x_6 + 1x_7 + 1x_8 + 1x_9 + 1x_{10}}{1+1+1+1+1+1+1+1+1+1}$$

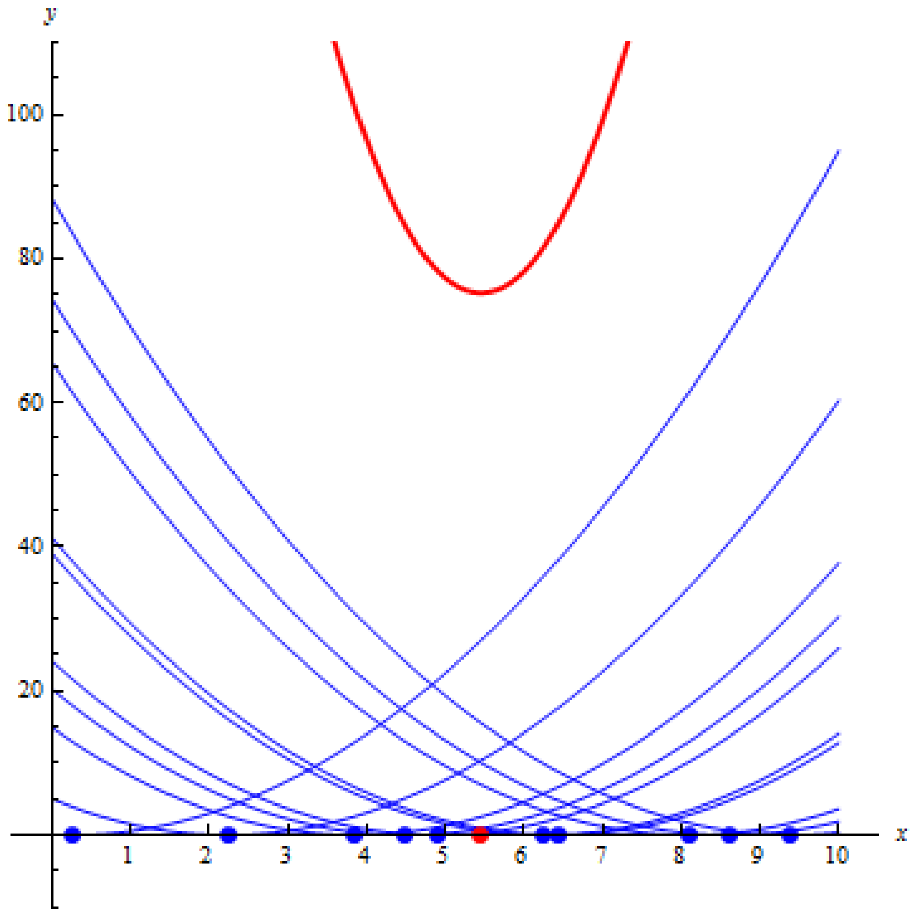
$$x = \frac{x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10}}{10}$$

Sample Mean

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

Sample Mean

$$\bar{x} = 5.45134$$

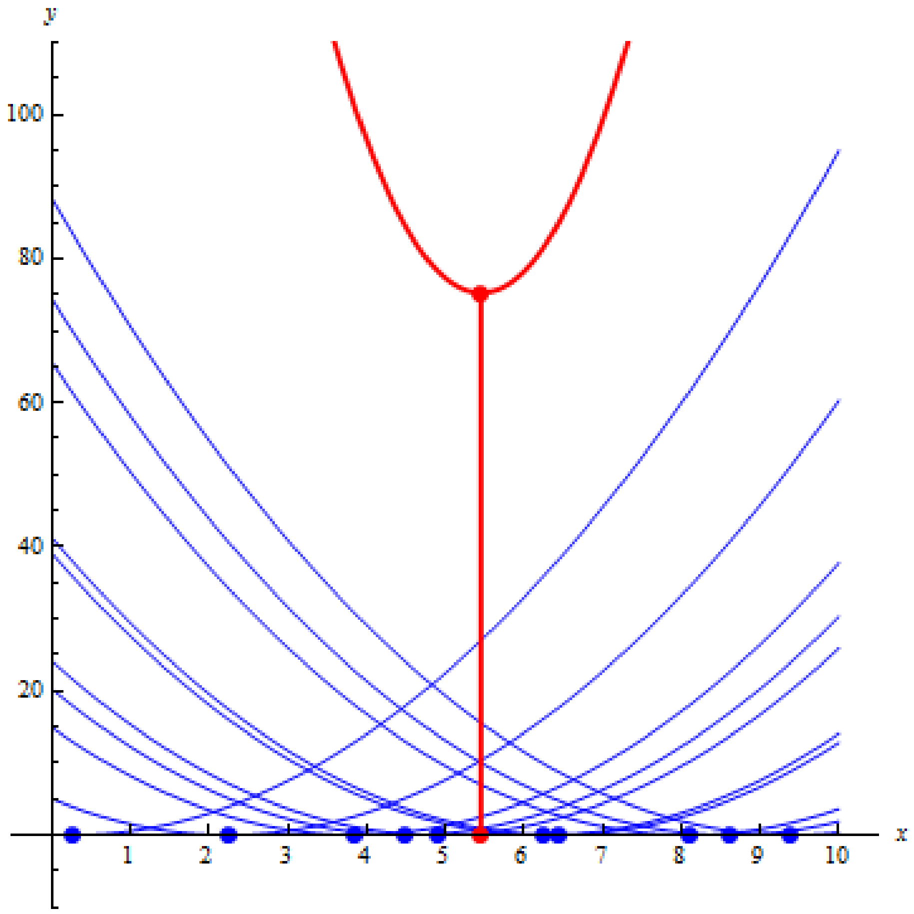


Sum of Parabolas: Sum of Squared Differences

$$y = \sum_{i=1}^{n} w_i \left(x - x_i \right)^2$$

Let $x = \overline{x}$ and $w_i = 1$, then

$$y = \sum_{i=1}^{n} \left(\overline{x} - x_i \right)^2$$



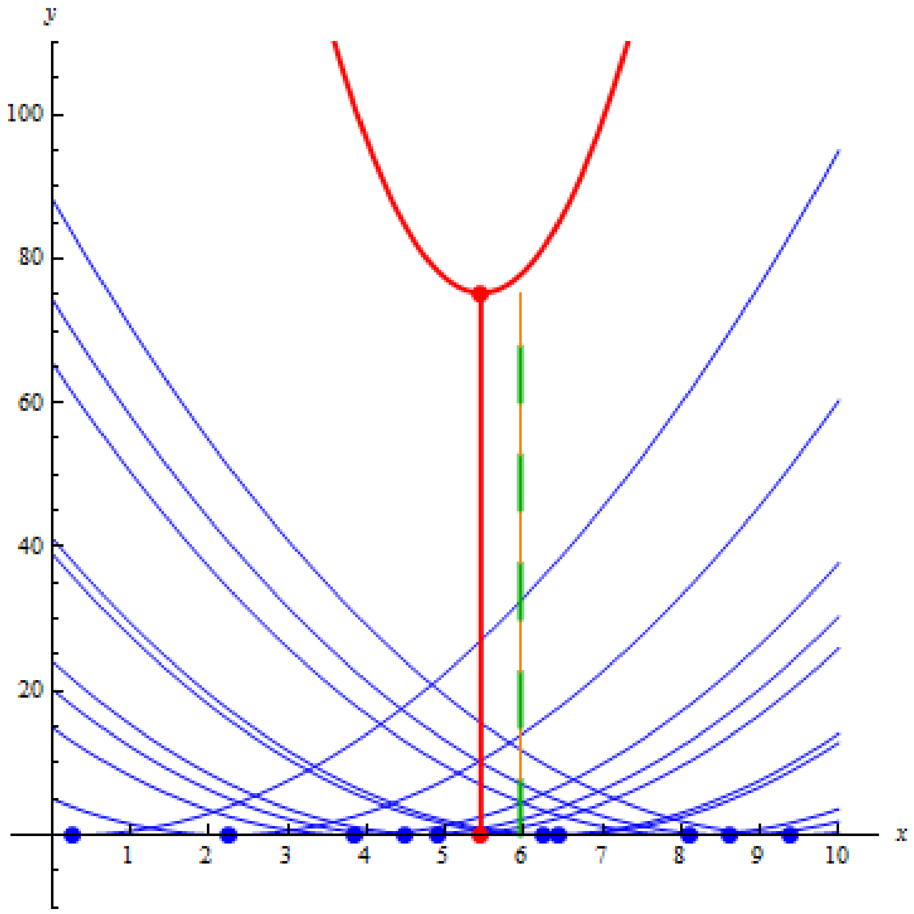
Sum of Parabolas: Biased Sample Variance

$$y = \sum_{i=1}^{n} w_i \left(x - x_i \right)^2$$

$$\text{Let } x = \bar{x} \text{ and } w_i = \frac{1}{n}, \text{ then}$$

$$y = \frac{1}{n} \sum_{i=1}^{n} \left(\bar{x} - x_i \right)^2$$

$$s_{\text{biased}}^2 = 7.52297$$



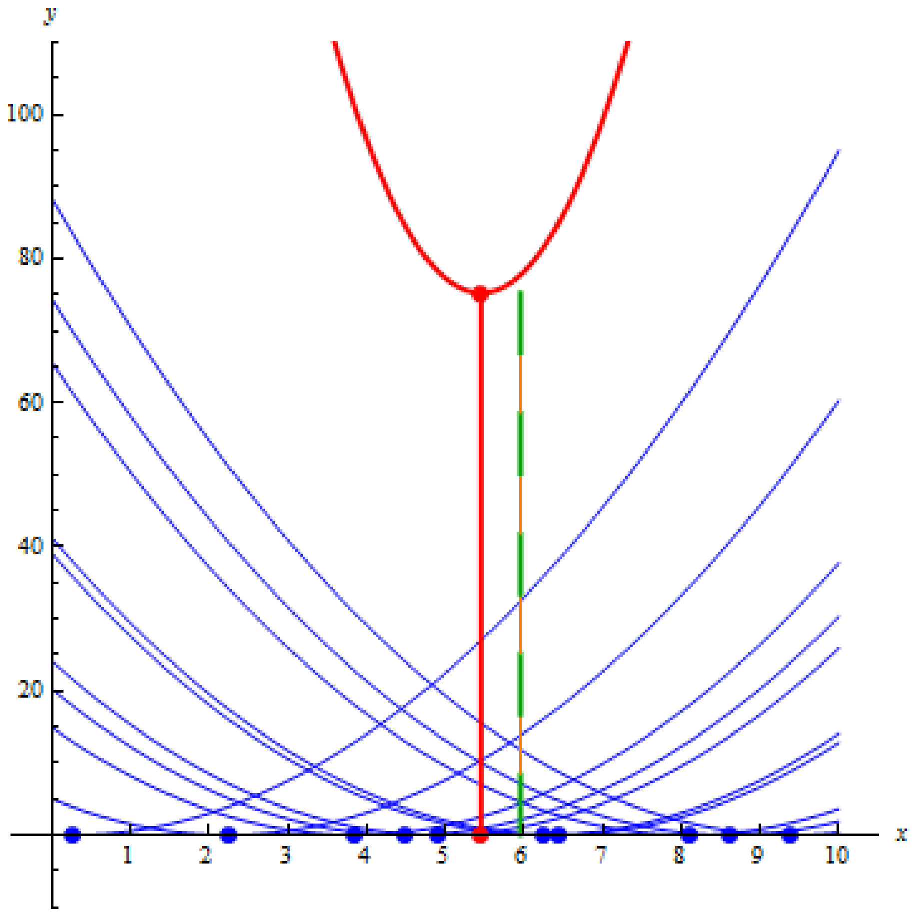
Sum of Parabolas: Unbiased Sample Variance

$$y = \sum_{i=1}^{n} w_i \left(x - x_i\right)^2$$

Let $x = \bar{x}$ and $w_i = \dfrac{1}{n-1}$, then

$$y = \frac{1}{n-1}\sum_{i=1}^{n}\left(\bar{x} - x_i\right)^2$$

$$s_{\text{unbiased}}^2 = 8.35885$$



Biased and Unbiased Sample Variance

$$y = \sum_{i=1}^{n} w_i \left(x - x_i \right)^2$$

If $x = \bar{x}$ and $w_i = \frac{1}{n}$, then

$$s^2{}_{\text{biased}} = \frac{1}{n} \sum_{i=1}^{n} \left(x_i - \bar{x} \right)^2$$

If $x = \bar{x}$ and $w_i = \frac{1}{n-1}$, then

$$s^2{}_{\text{unbiased}} = \frac{1}{n-1} \sum_{i=1}^{n} \left(x_i - \bar{x} \right)^2$$

Sample Standard Deviation

- Biased:

$$\sqrt{s^2_{\text{biased}}} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$$

$$s_{\text{biased}} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$$

- Unbiased:

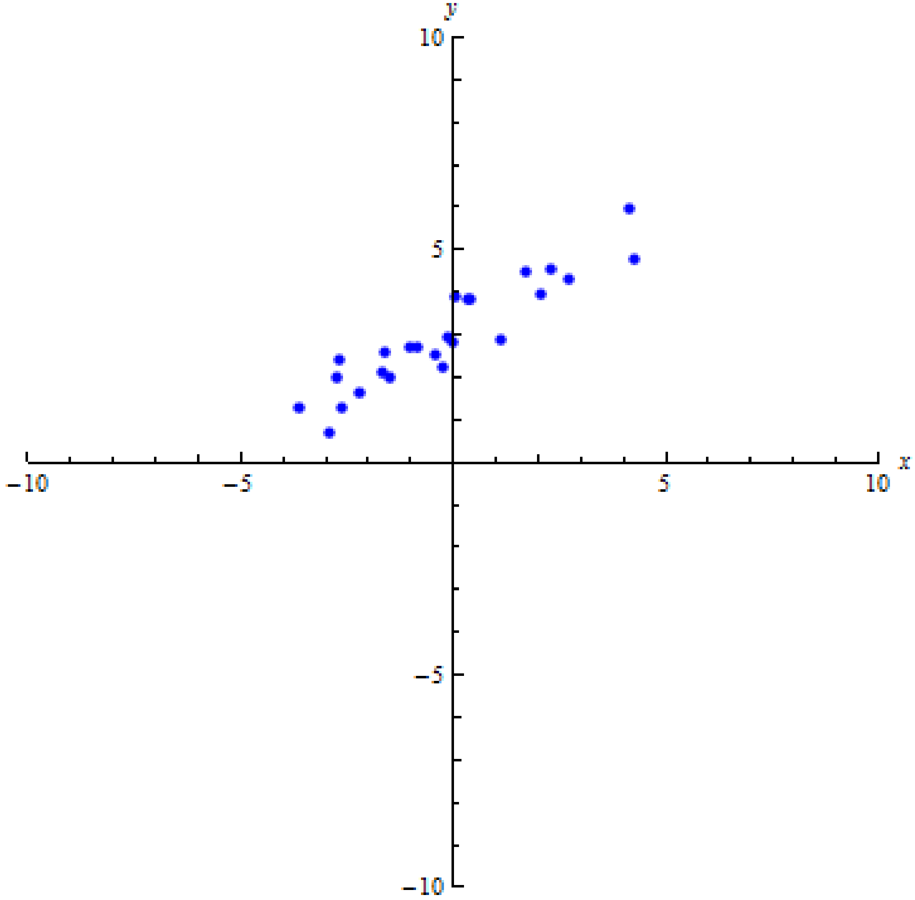
$$\sqrt{s^2_{\text{unbiased}}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$$

$$s_{\text{unbiased}} = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$$

Data

- Data:

```
{
    {-2.70238,2.43155},
    {-2.18612,1.63634},
    {0.405141,3.84976},
    {-1.03072,2.7144},
    {1.10033,2.89639},
    {-1.63906,2.56916},
    {2.27983,4.57127},
    {-0.836348,2.70824},
    {-2.90988,0.685828},
    {-0.104817,2.95222},
    {-0.226538,2.24849},
    {-2.64364,1.28981},
    {-0.00953108,2.84022},
    {0.336282,3.86626},
    {4.12633,5.94993},
    {1.70053,4.51368},
    {-1.4793,1.99986},
    {0.0467884,3.91811},
    {-1.70285,2.12758},
    {-3.61035,1.26436},
    {2.08504,3.94459},
    {4.23512,4.80965},
    {2.70993,4.30984},
    {-2.72741,1.97363},
    {-0.418723,2.55797}
}
```



Data

- Data:

```
{
	{-2.70238,2.43155},
	{-2.18612,1.63634},
	{0.405141,3.84976},
	{-1.03072,2.7144},
	{1.10033,2.89639},
	{-1.63906,2.56916},
	{2.27983,4.57127},
	{-0.836348,2.70824},
	{-2.90988,0.685828},
	{-0.104817,2.95222},
	{-0.226538,2.24849},
	{-2.64364,1.28981},
	{-0.00953108,2.84022},
	{0.336282,3.86626},
	{4.12633,5.94993},
	{1.70053,4.51368},
	{-1.4793,1.99986},
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	{2.08504,3.94459},
	{4.23512,4.80965},
	{2.70993,4.30984},
	{-2.72741,1.97363},
	{-0.418723,2.55797}
}
```
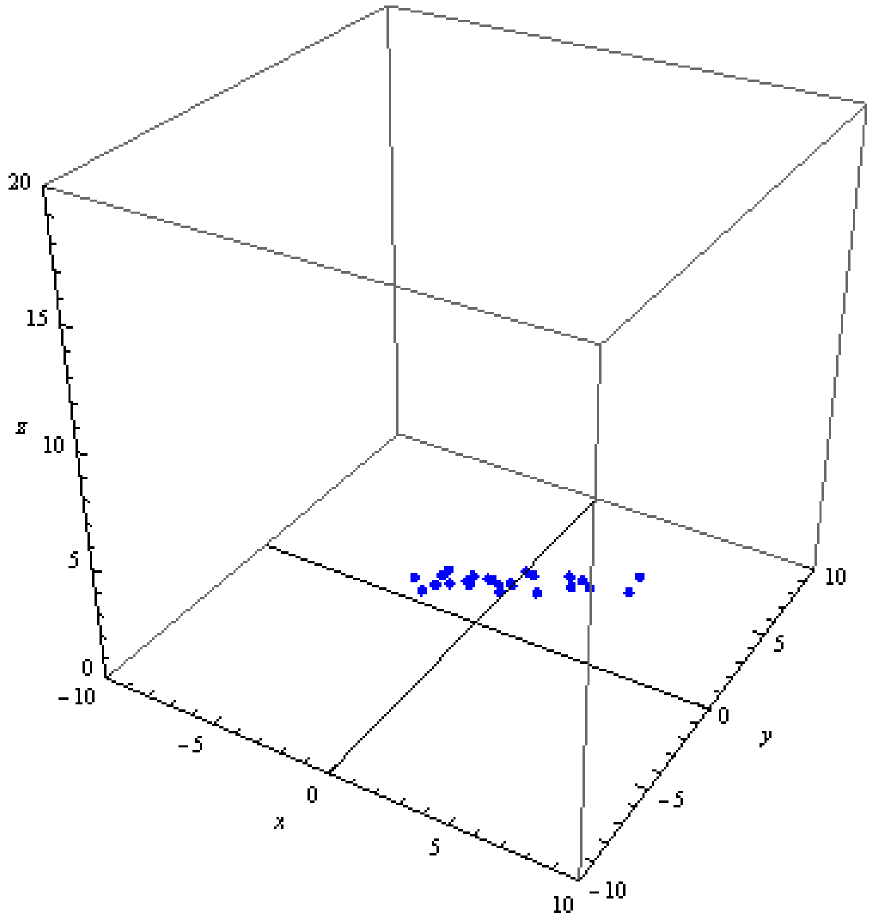
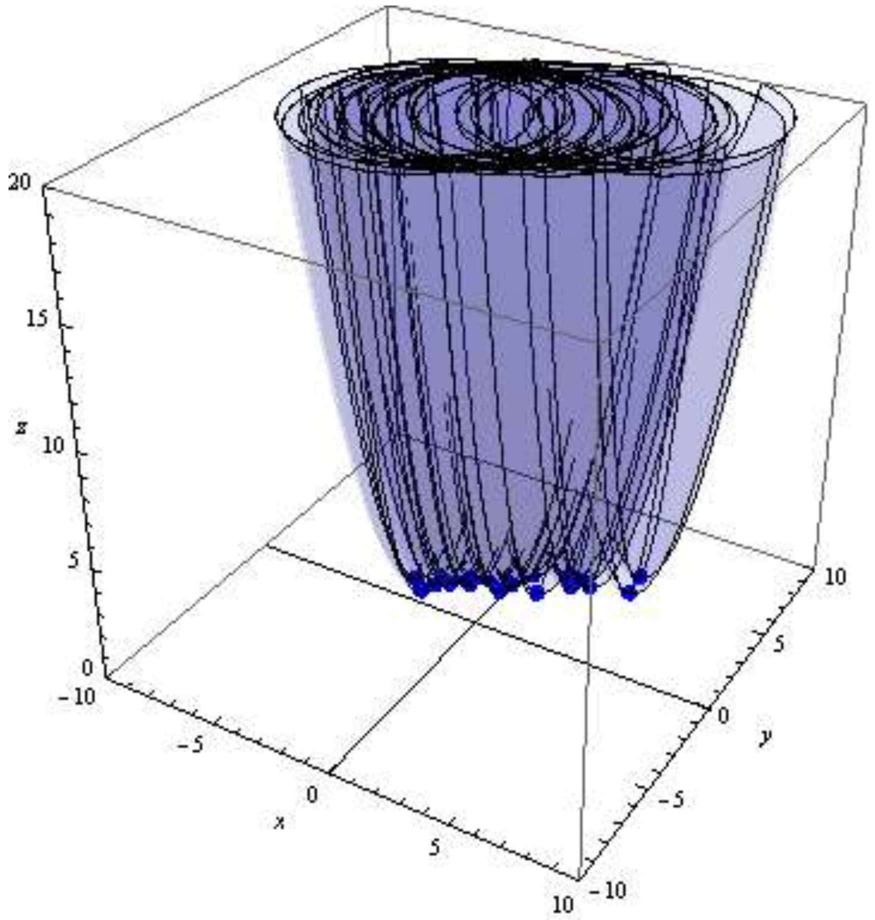



2D Parabolas

- 2D Parabola:

$$z = (x - x_0)^2 + (y - y_0)^2$$



Sum of 2D Parabolas

$$f(x,y) = \sum_{i=1}^{n}\left((x - x_i)^2 + (y - y_i)^2\right) = \sum_{i=1}^{n}(x - x_i)^2 + \sum_{i=1}^{n}(y - y_i)^2$$

$$\frac{\partial f}{\partial x} = \sum_{i=1}^{n} 2(x - x_i) \qquad \frac{\partial^2 f}{\partial x^2} = \sum_{i=1}^{n} 2 = 2n \qquad \frac{\partial^2 f}{\partial x \partial y} = 0$$

$$\frac{\partial f}{\partial y} = \sum_{i=1}^{n} 2(y - y_i) \qquad \frac{\partial^2 f}{\partial y \partial x} = 0 \qquad \frac{\partial^2 f}{\partial y^2} = \sum_{i=1}^{n} 2 = 2n$$

Extrema: First Derivative

$$\frac{\partial f}{\partial x} = 0 \text{ and } \frac{\partial f}{\partial x} = \sum_{i=1}^{n} 2\left(x - x_i\right)$$

$$\sum_{i=1}^{n} 2\left(x - x_i\right) = 0 \quad \sum_{i=1}^{n} x - \sum_{i=1}^{n} x_i = 0 \quad nx = \sum_{i=1}^{n} x_i$$

$$\sum_{i=1}^{n} \left(x - x_i\right) = 0 \quad \sum_{i=1}^{n} x = \sum_{i=1}^{n} x_i \quad x = \frac{\sum_{i=1}^{n} x_i}{n}$$

Extrema: First Derivative

$$\frac{\partial f}{\partial y} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} = \sum_{i=1}^{n} 2\left(y - y_i\right)$$

$$\sum_{i=1}^{n} 2\left(y - y_i\right) = 0 \qquad \sum_{i=1}^{n} y - \sum_{i=1}^{n} y_i = 0 \qquad ny = \sum_{i=1}^{n} y_i$$

$$\sum_{i=1}^{n} \left(y - y_i\right) = 0 \qquad \sum_{i=1}^{n} y = \sum_{i=1}^{n} y_i \qquad y = \frac{\sum_{i=1}^{n} y_i}{n}$$

Extrema: Second Derivative (Hessian Matrix)

Minima if:

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix}$$

$$0 < \left| \left(\frac{\partial^2 f}{\partial x^2} \right) \right| \text{ and } 0 < \left| \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \right|$$

$$0 < \frac{\partial^2 f}{\partial x^2} \text{ and } 0 < \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x \partial y}\frac{\partial^2 f}{\partial y \partial x}$$

Extrema: Second Derivative (Hessian Matrix)

$$\begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}$$

Minima if:

$$0 < \left|\left(2n\right)\right| \text{ and } 0 < \left|\begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}\right|$$

$$0 < 2n \text{ and } 0 < (2n)(2n) - (0)(0)$$

$$0 < 2n \text{ and } 0 < (2n)^2$$

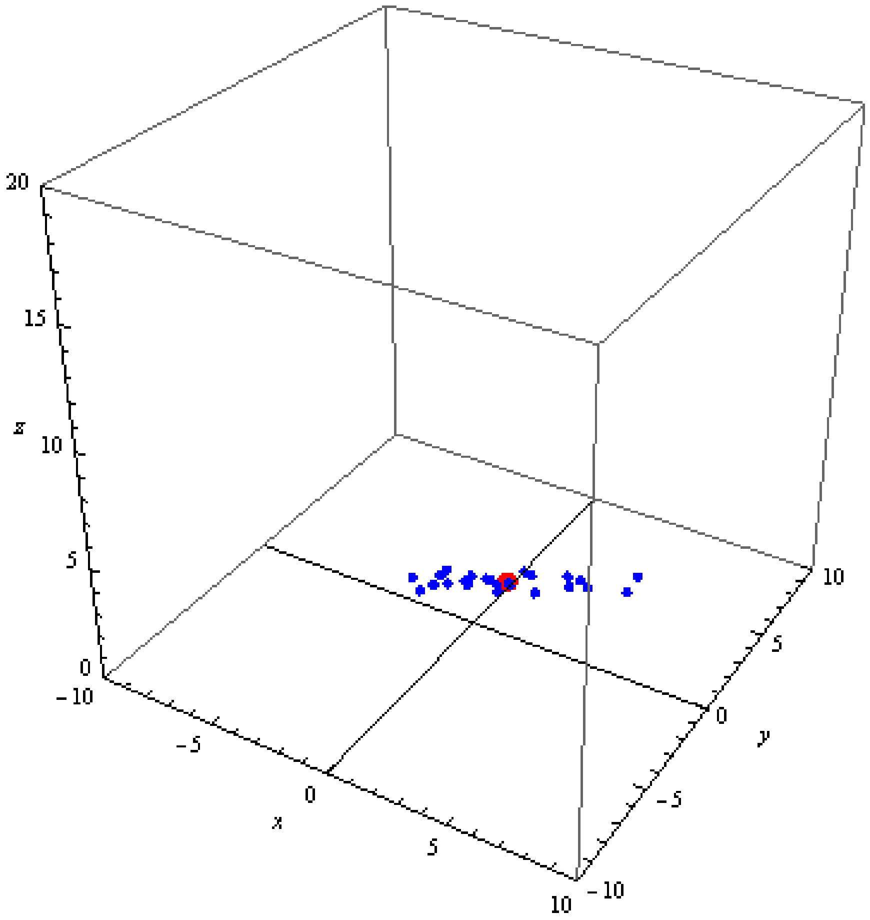
Minima: Sample Mean

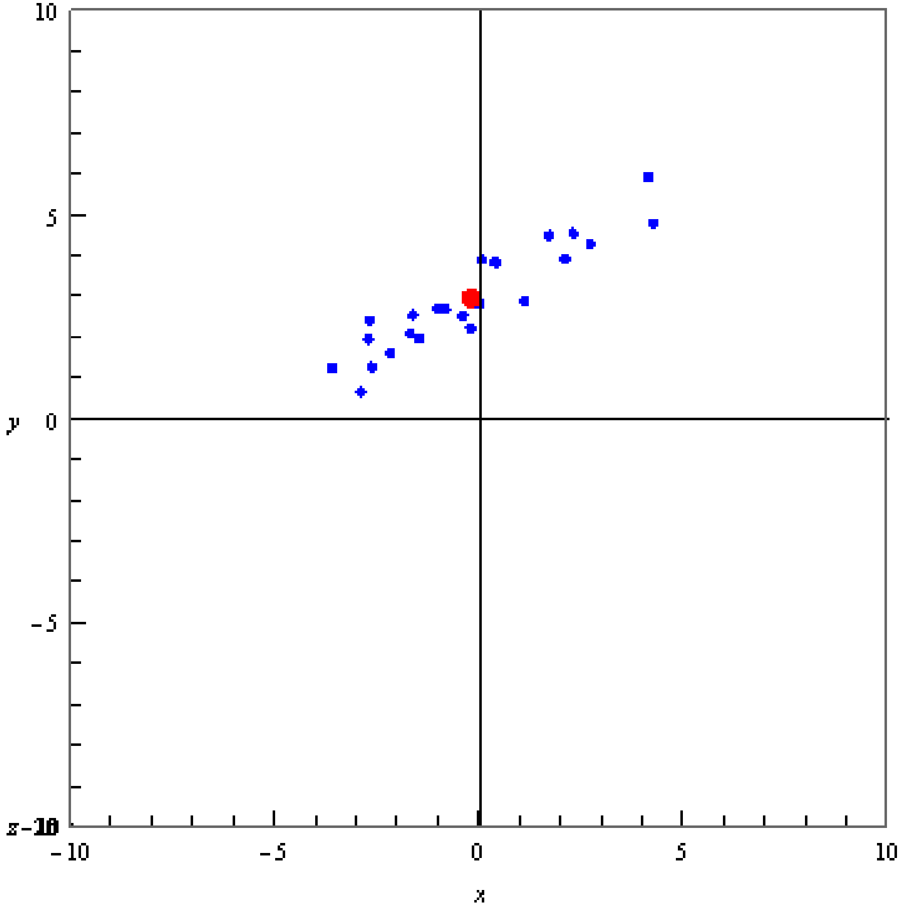
Therefore, we have a minima at:

$$x = \frac{\sum_{i=1}^{n} x_i}{n} \qquad y = \frac{\sum_{i=1}^{n} y_i}{n}$$

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad \bar{y} = \frac{\sum_{i=1}^{n} y_i}{n}$$

Sample Mean



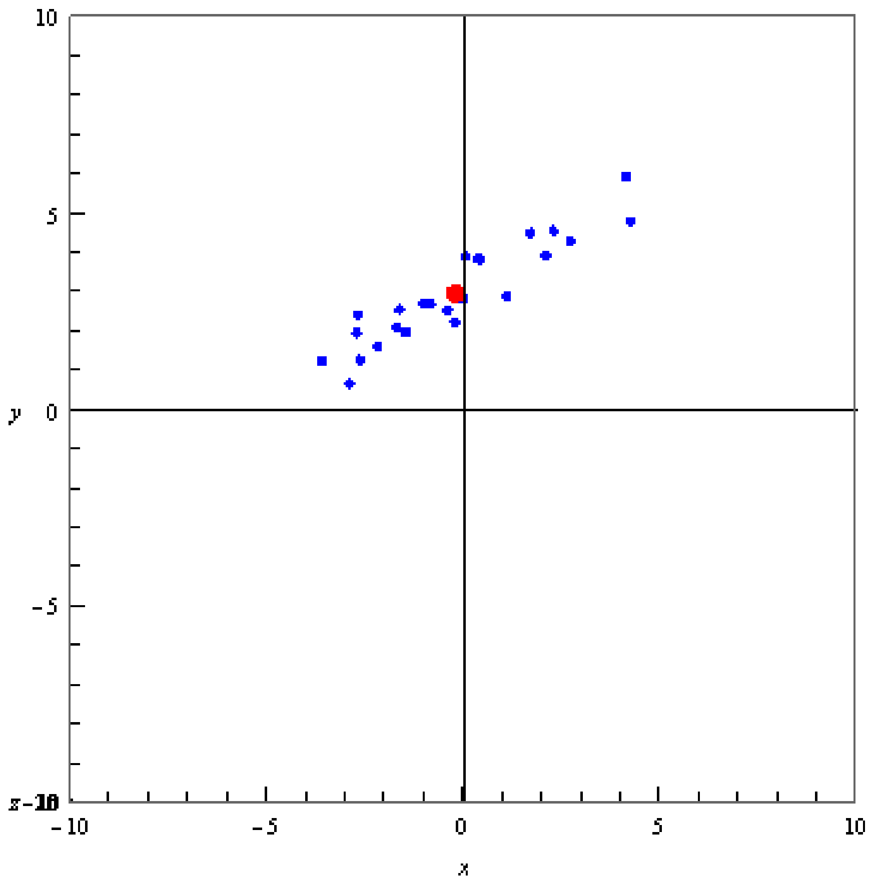


Sample Mean

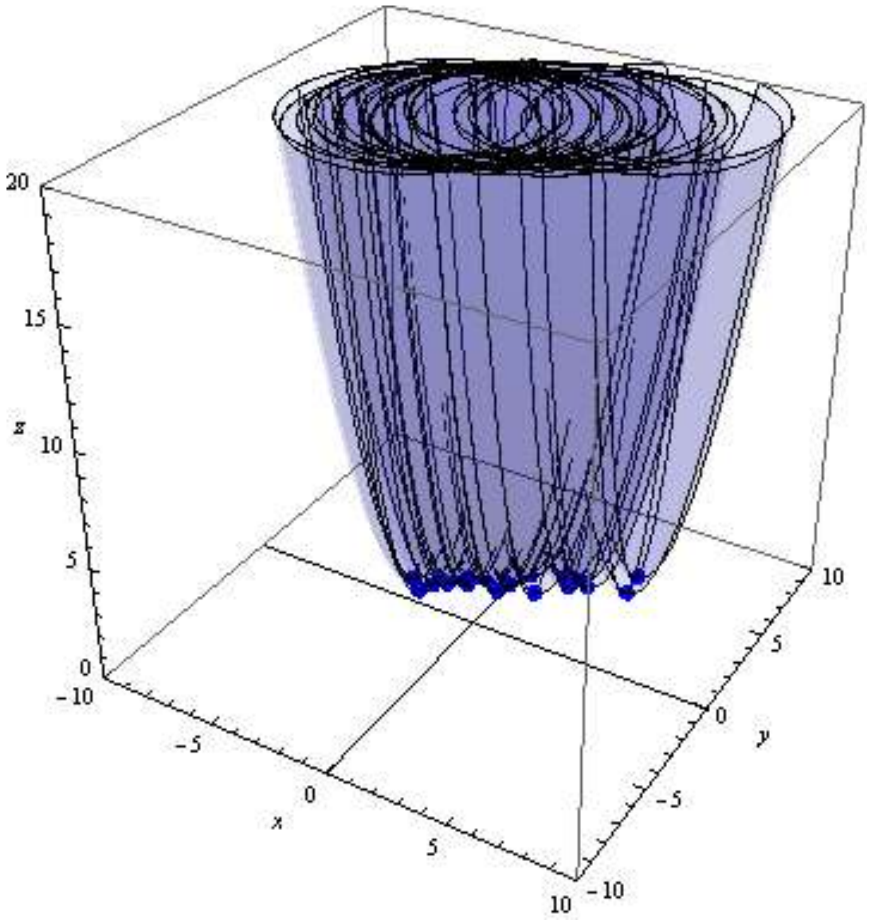
$$\bar{x} = -0.208094$$

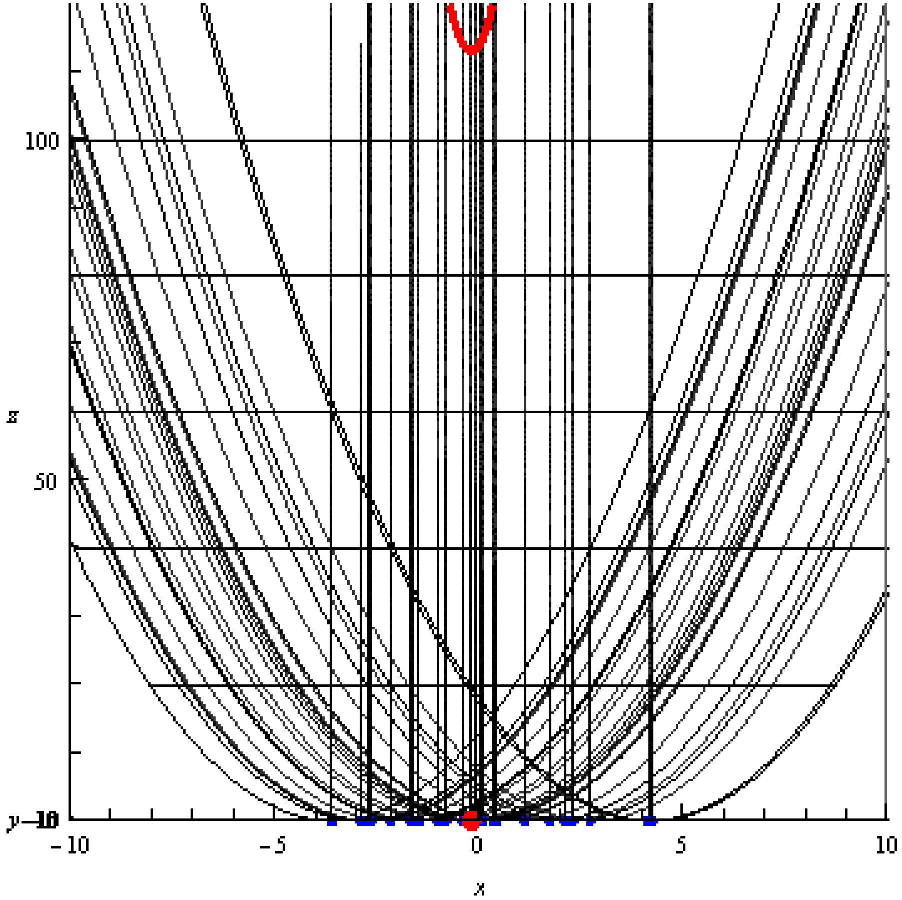
$$\bar{y} = 2.98517$$

$$(-0.208094, 2.98517)$$

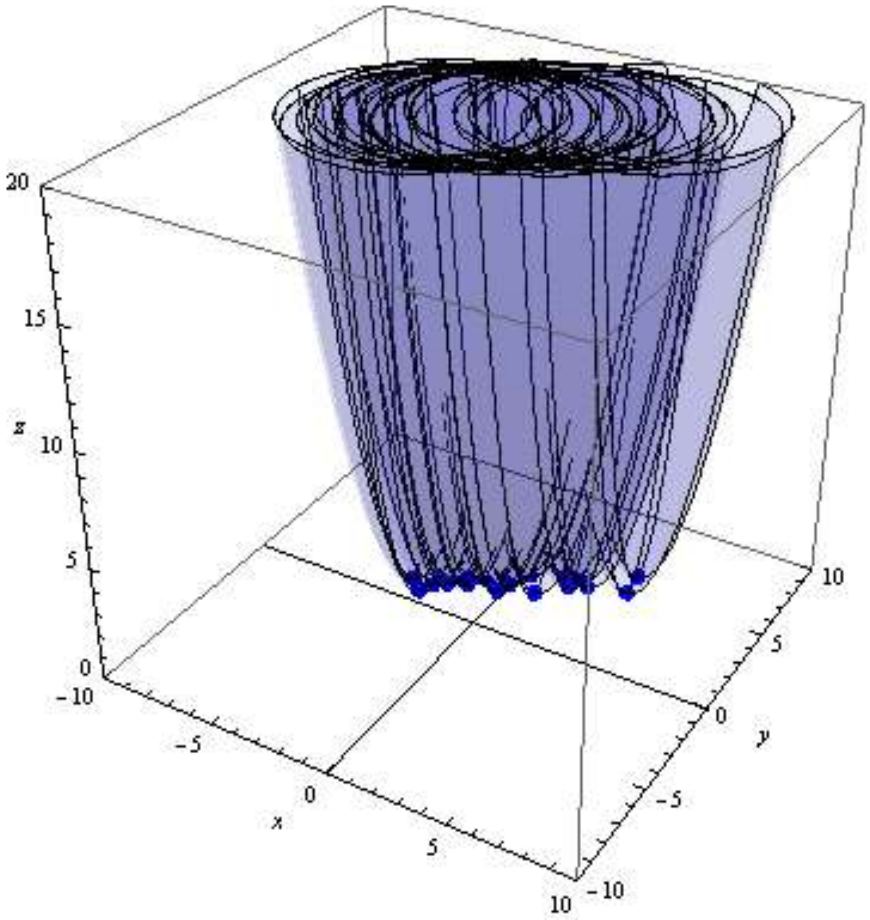


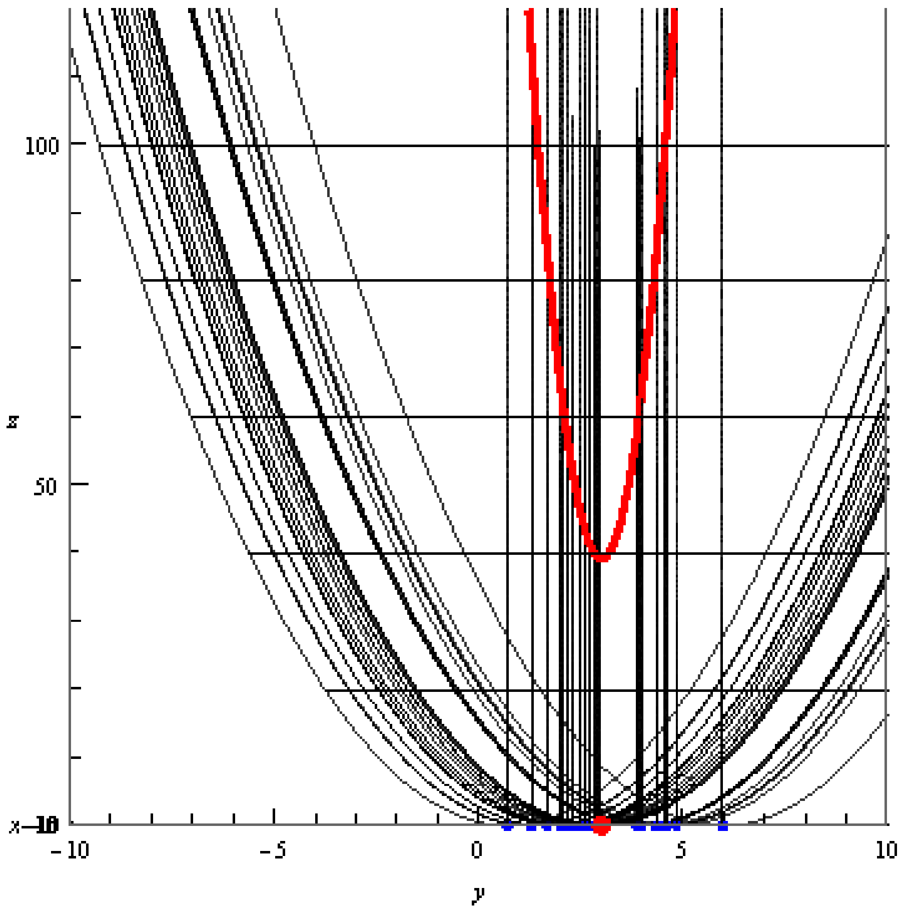
Sum of Parabolas: x





Sum of Parabolas: y





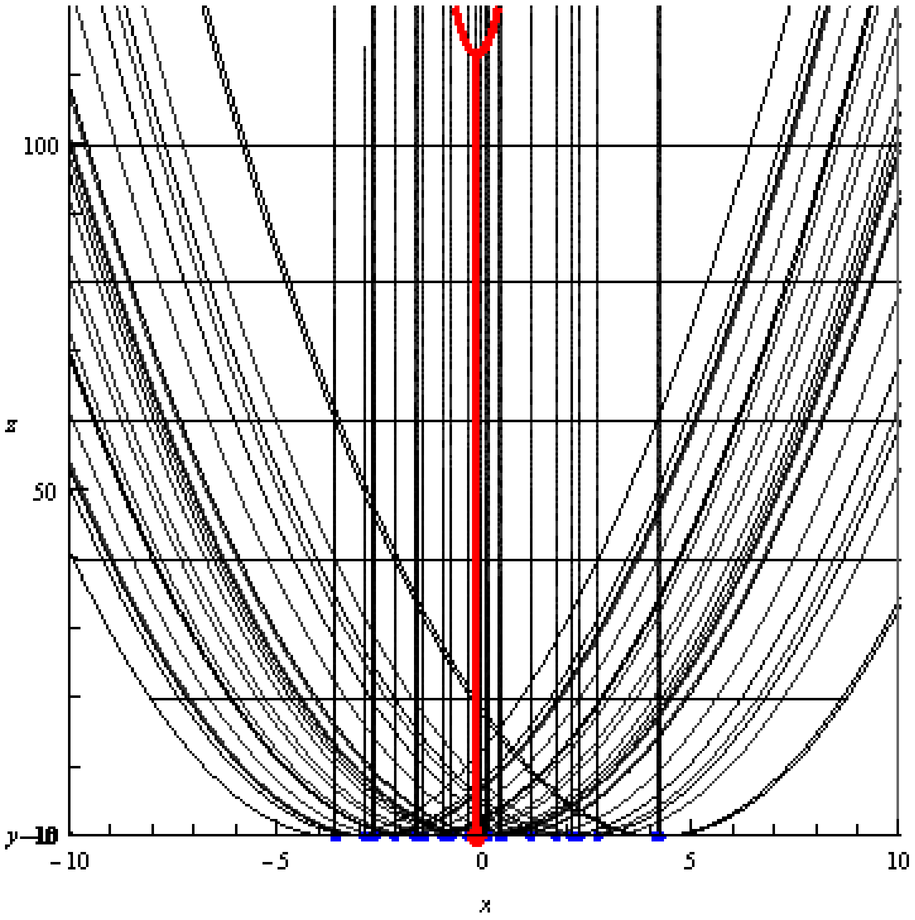
Sum of Squared Differences: x

$$z_{xx} = \sum_{i=1}^{n} \left(x - x_i \right)^2$$

Let $x = \bar{x}$, then

$$z_{xx} = \sum_{i=1}^{n} \left(\bar{x} - x_i \right)^2$$

$$z_{xx} = 112.937$$



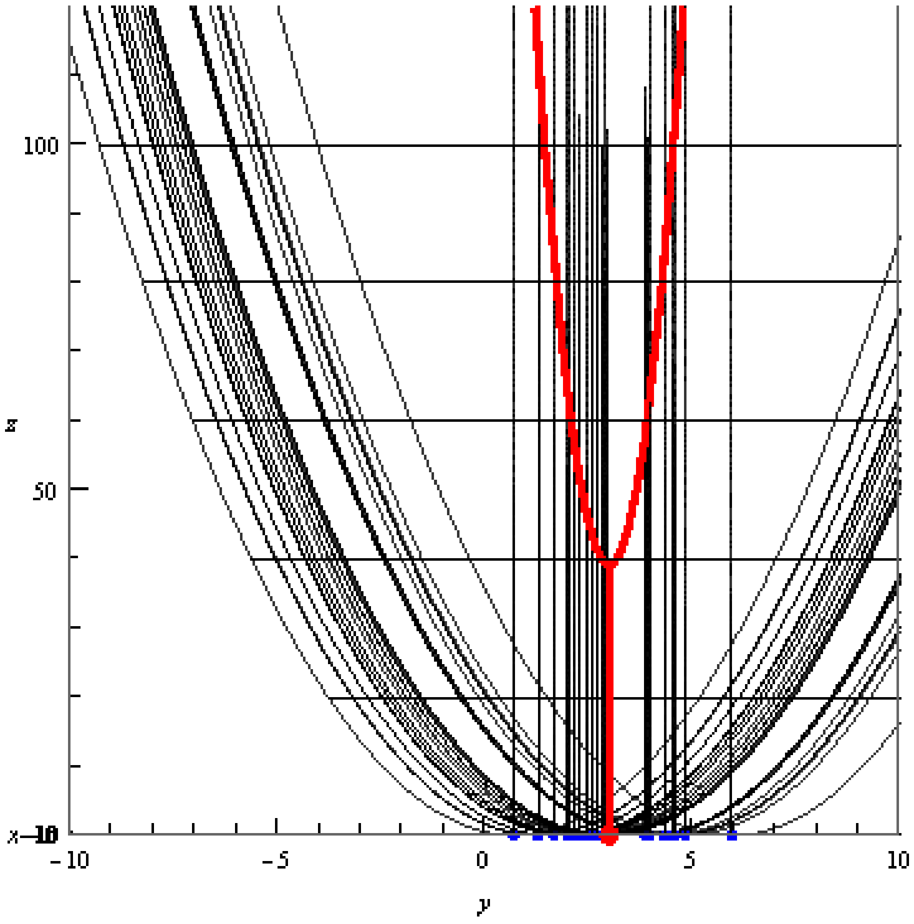
Sum of Squared Differences: y

$$z_{yy} = \sum_{i=1}^{n} \left(y - y_i \right)^2$$

Let $y = \bar{y}$, then

$$z_{yy} = \sum_{i=1}^{n} \left(\bar{y} - y_i \right)^2$$

$$z_{yy} = 39.096$$



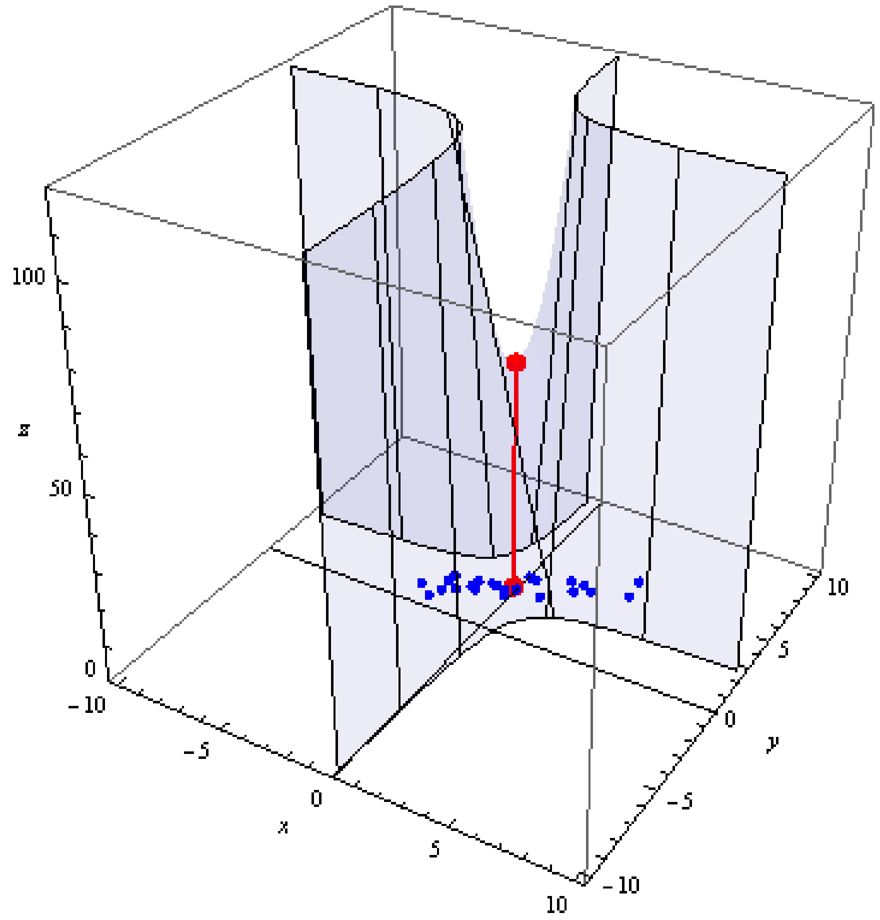
Sum of the Product of Differences

$$z_{xy} = \sum_{i=1}^{n} (x - x_i)(y - y_i)$$

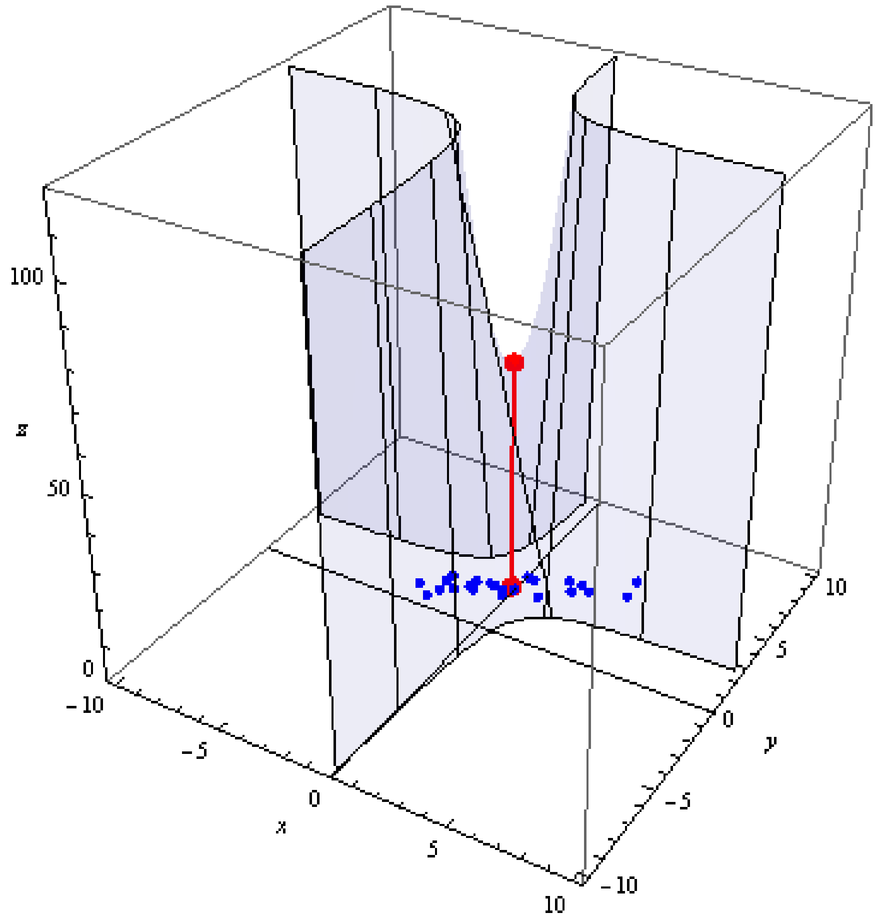
Let $x = \bar{x}$ and $y = \bar{y}$, then

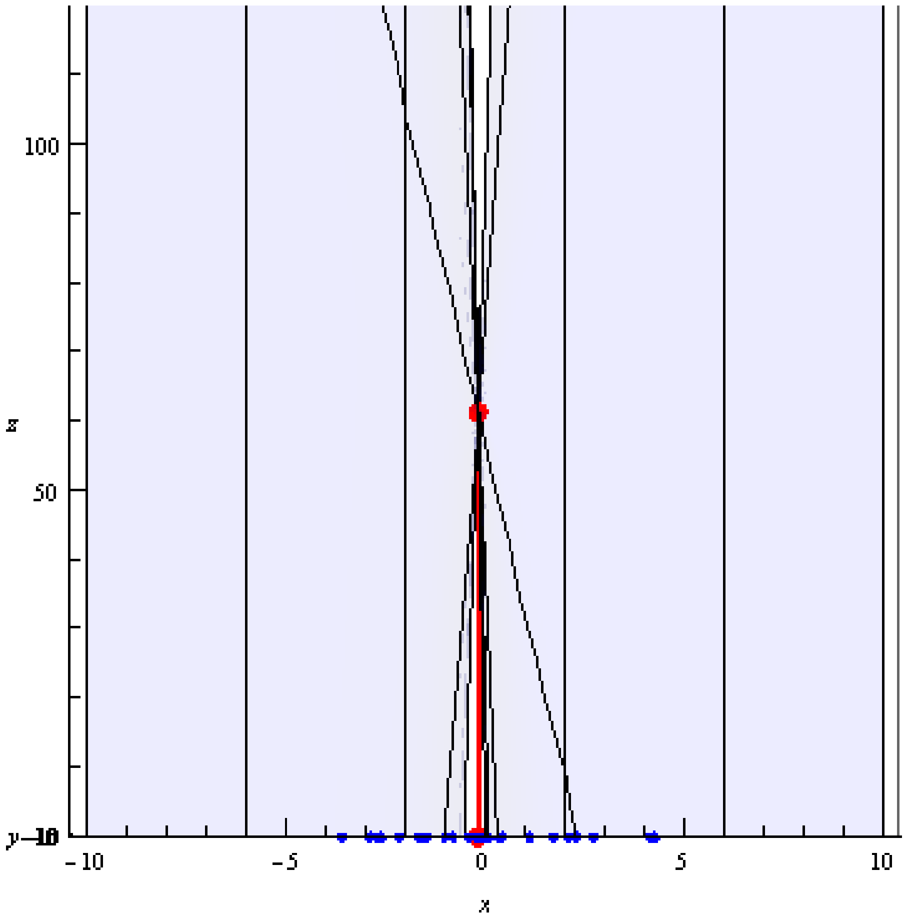
$$z_{xy} = \sum_{i=1}^{n} (\bar{x} - x_i)(\bar{y} - y_i)$$

$$z_{xy} = 61.4087$$

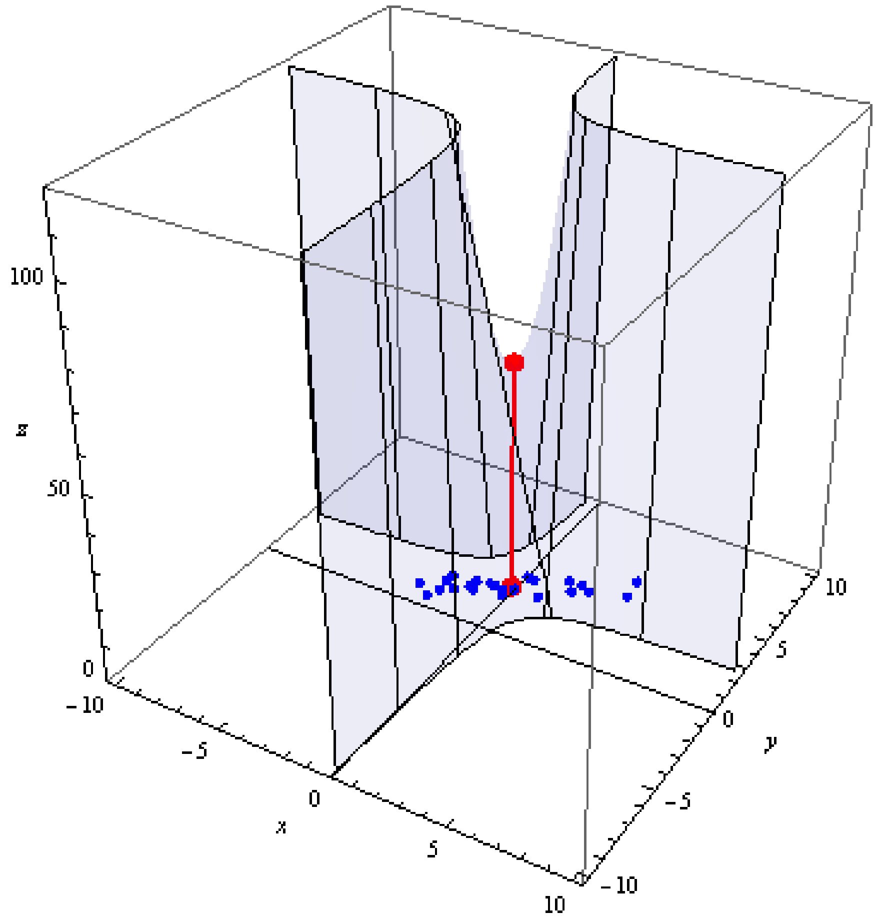


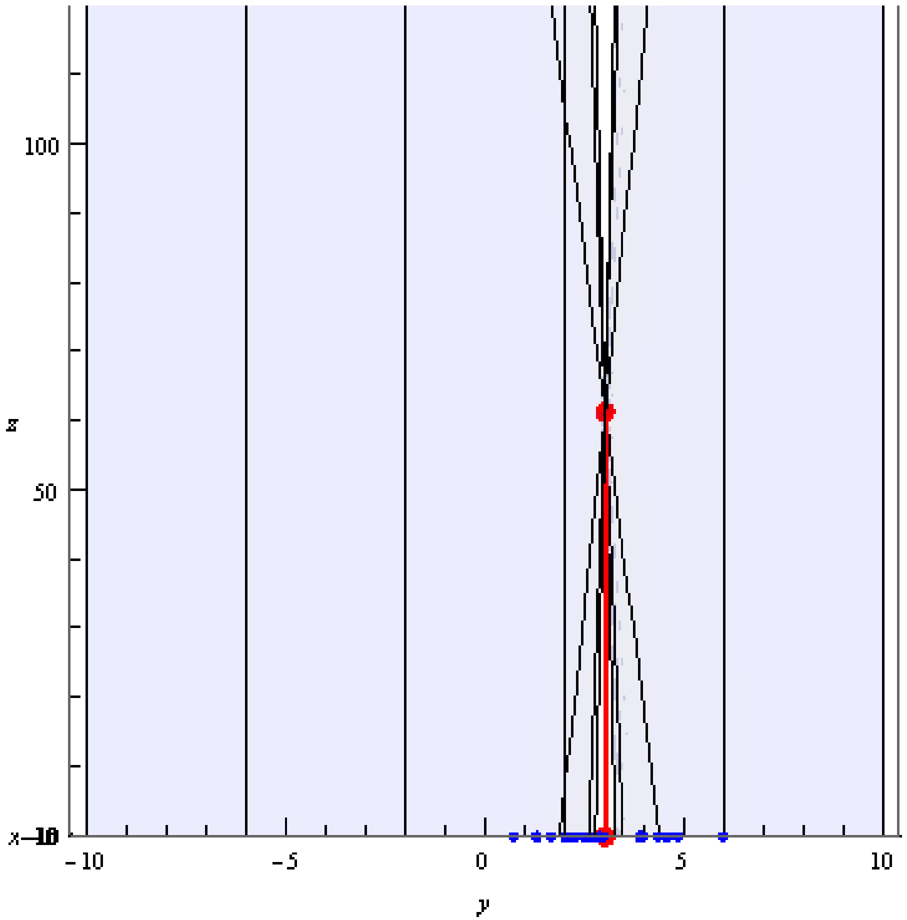
Sum of the Product of Differences: x





Sum of the Product of Differences: y





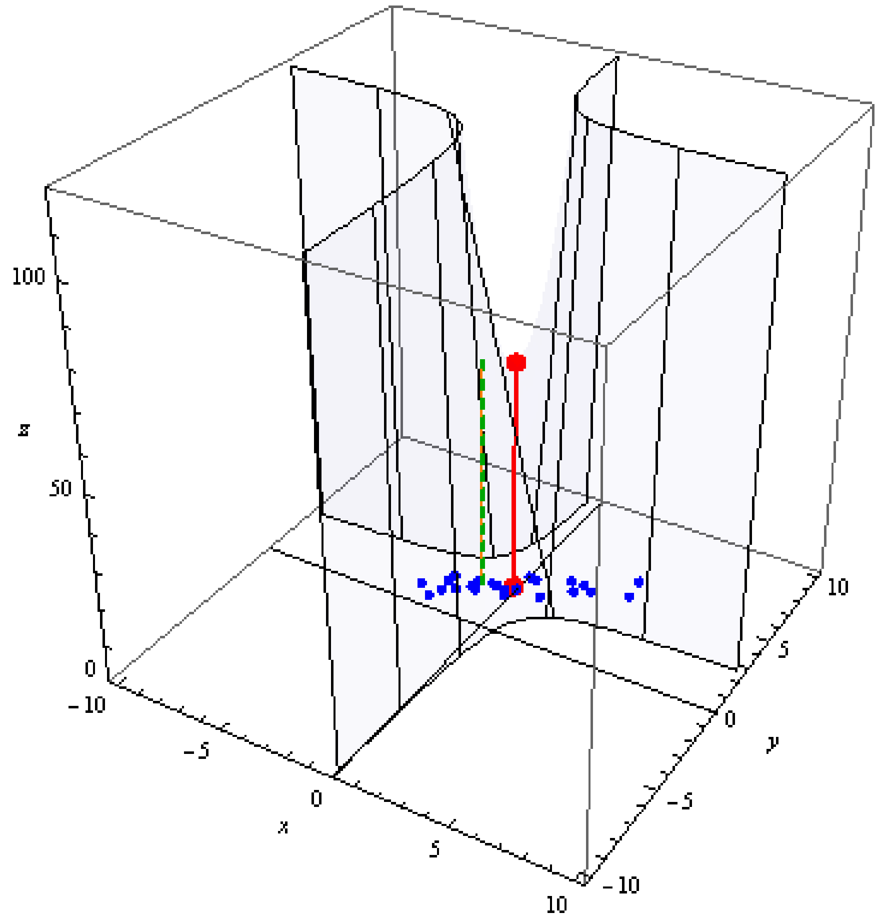
Sum of the Product of Differences: Biased Sample Covariance

$$z_{xy} = \sum_{i=1}^{n} w_i \left(x - x_i\right)\left(y - y_i\right)$$

Let $x = \bar{x}$, $y = \bar{y}$, and $w_i = \dfrac{1}{n}$, then

$$z_{xy} = \frac{1}{n}\sum_{i=1}^{n}\left(\bar{x} - x_i\right)\left(\bar{y} - y_i\right)$$

$$\operatorname{cov}_{\text{biased}}\left(x, y\right) = 2.45635$$



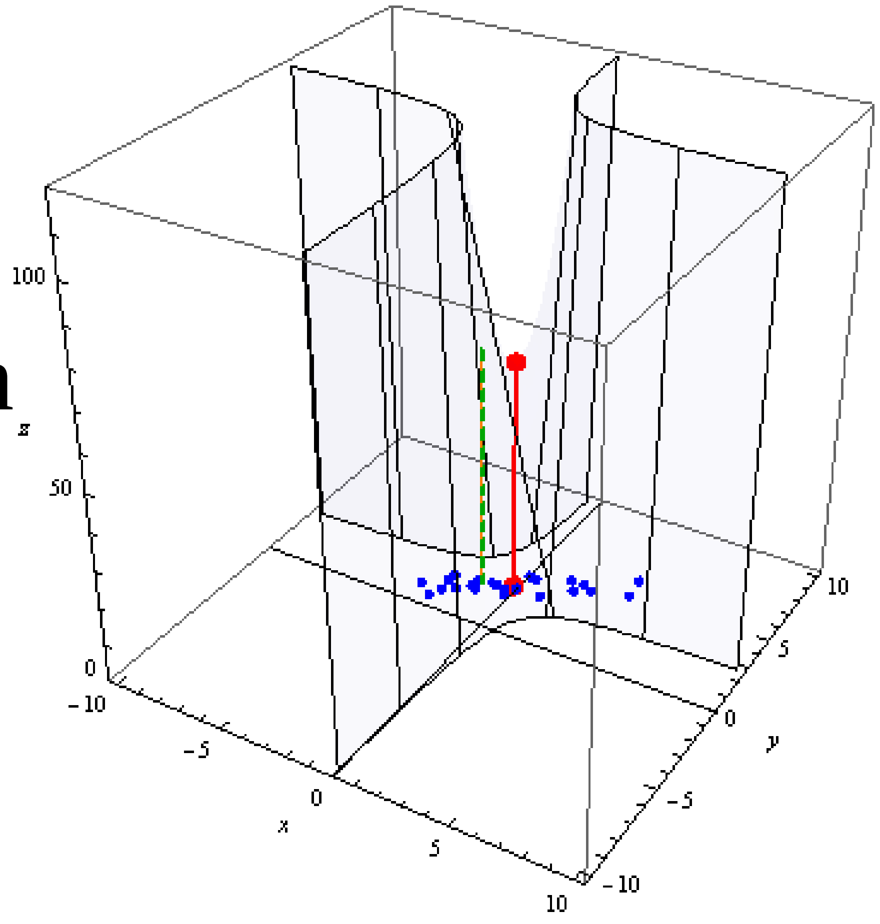
Sum of the Product of Differences: Unbiased Sample Covariance

$$z_{xy} = \sum_{i=1}^{n} w_i (x - x_i)(y - y_i)$$

Let $x = \bar{x}$, $y = \bar{y}$, and $w_i = \dfrac{1}{n-1}$, then

$$z_{xy} = \frac{1}{n-1}\sum_{i=1}^{n} (\bar{x} - x_i)(\bar{y} - y_i)$$

$$\text{cov}_{\text{unbiased}}(x, y) = 2.5587$$

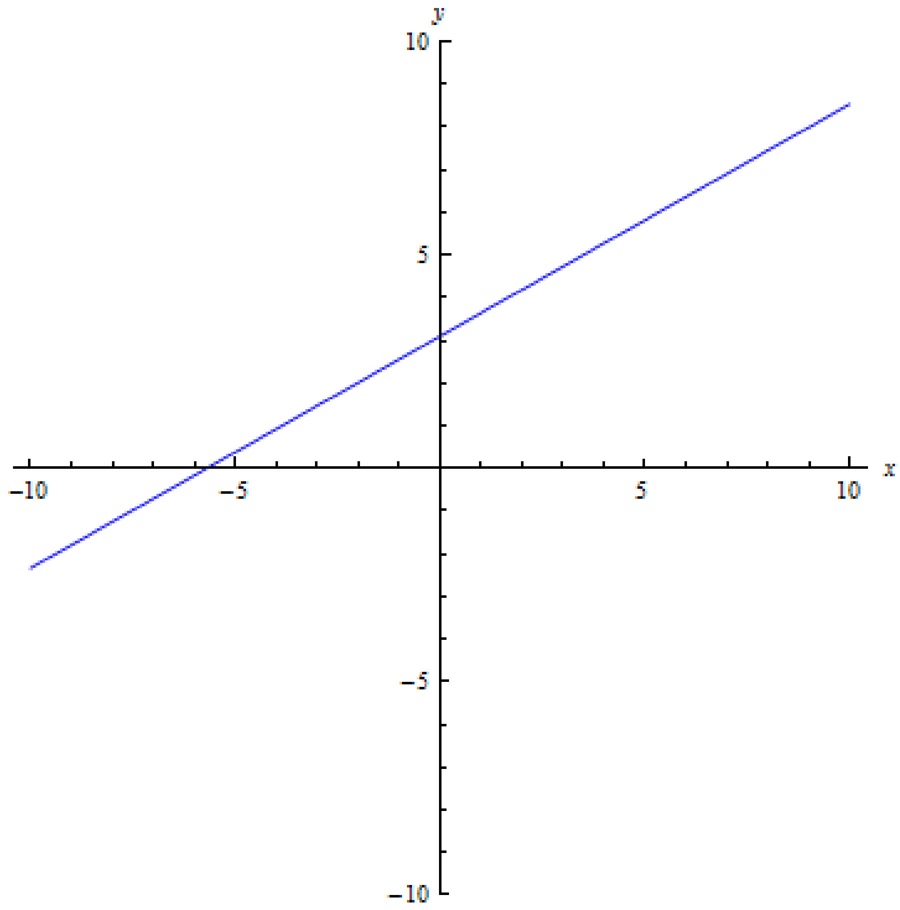


Line

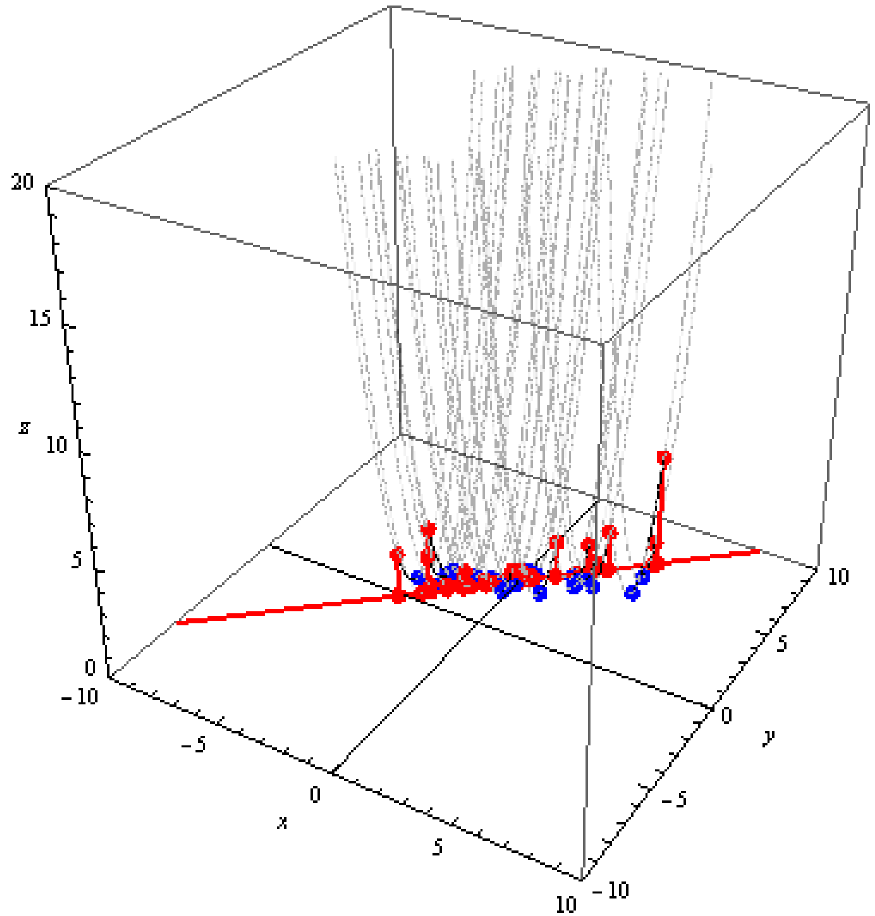
$$y = w\left(x - x_0\right) + y_0$$

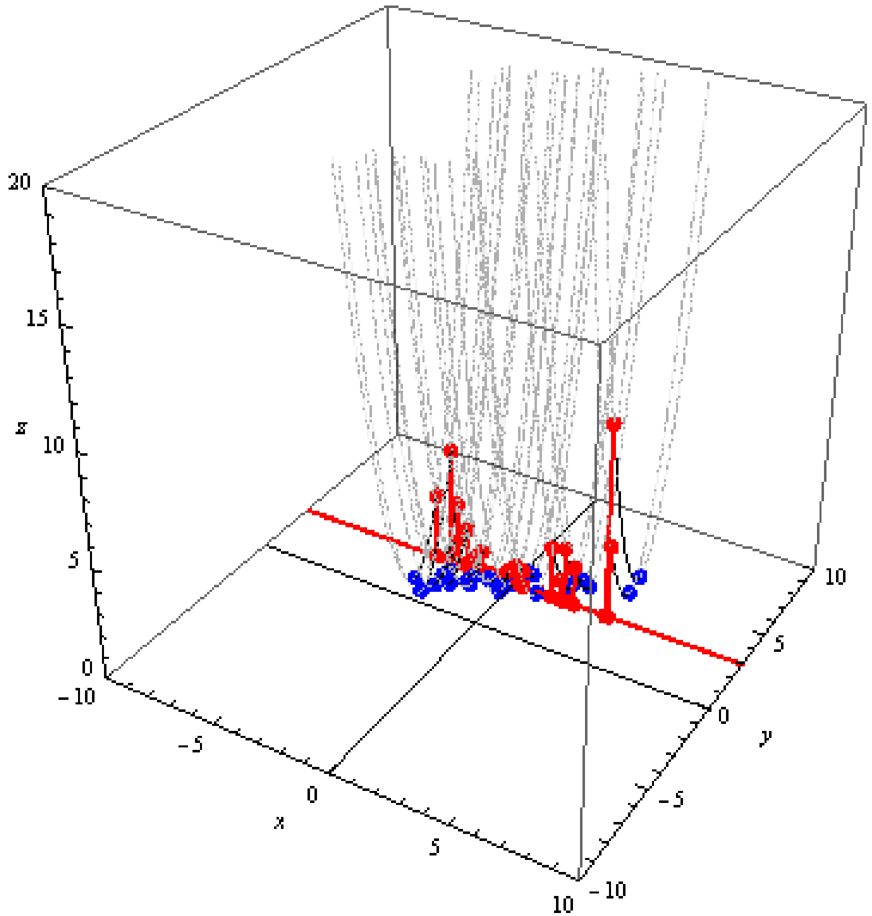
$$y = wx - wx_0 + y_0$$

$$y = \underbrace{\left(w\right)}_{m} x + \underbrace{\left(-wx_0 + y_0\right)}_{b}$$

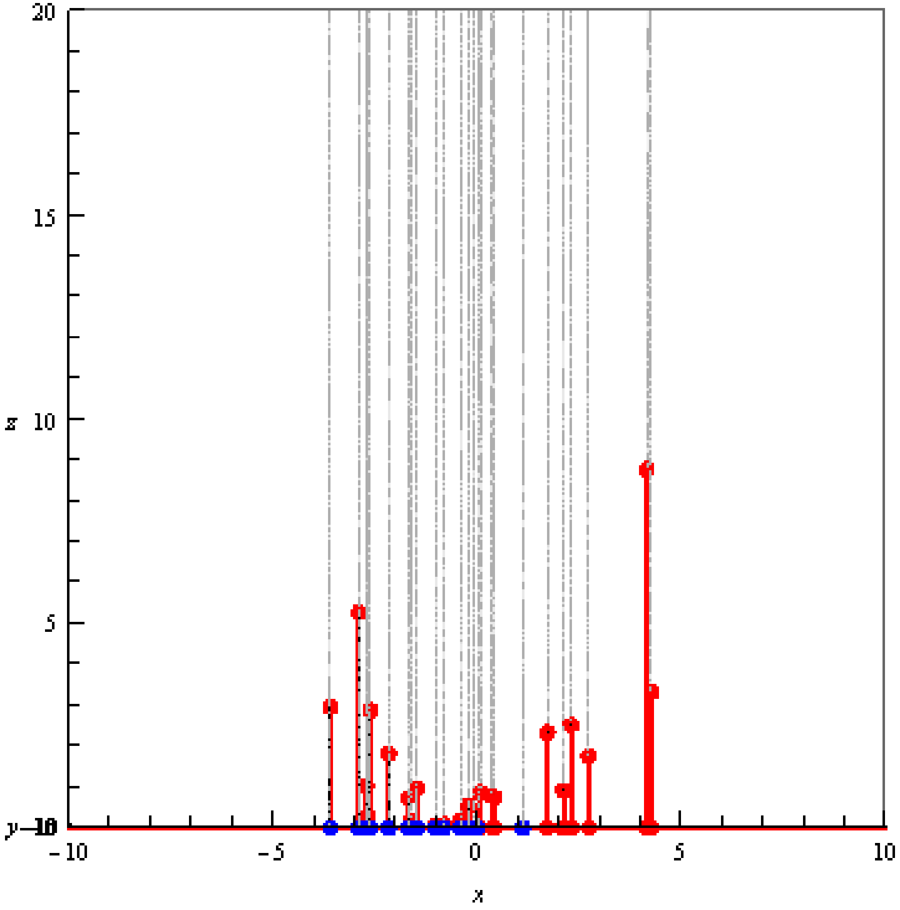


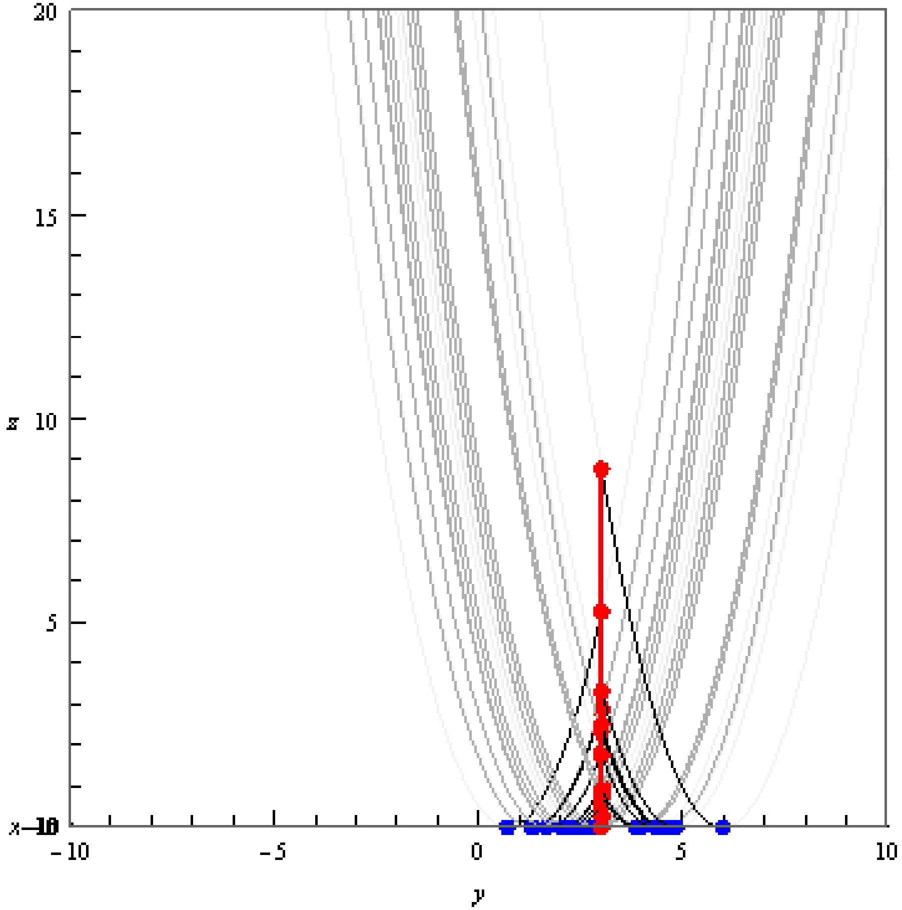
Linear Regression: Parabolas





Linear Regression: Parabolas





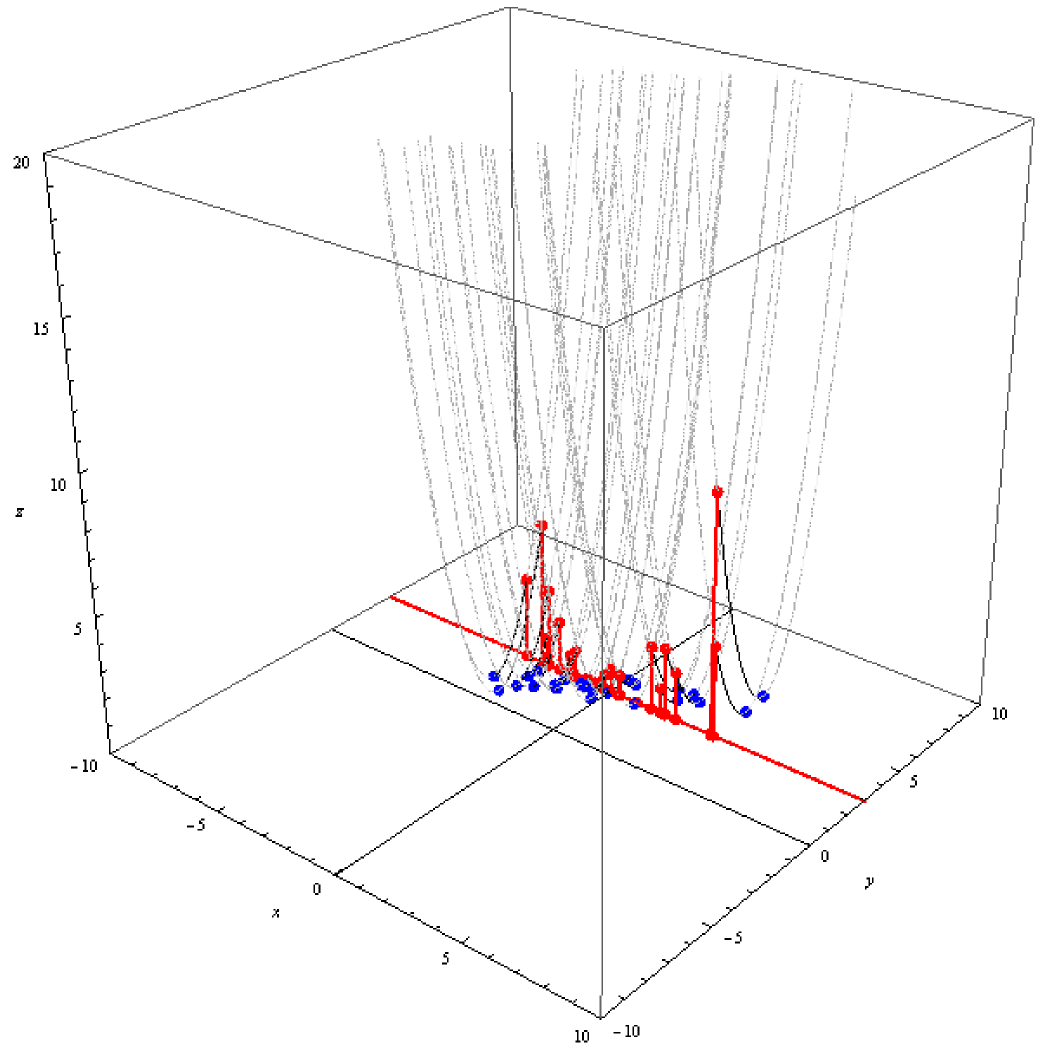
Linear Regression

$$y = w\left(x - \bar{x}\right) + \bar{y}$$

$$f\left(w\right) = \sum_{i=1}^{n}\left(\left(y_i\right) - \left(w\left(x_i - \bar{x}\right) + \bar{y}\right)\right)^2$$

$$= \sum_{i=1}^{n}\left(-\left(x_i - \bar{x}\right)w + \left(y_i - \bar{y}\right)\right)^2$$

Linear Regression



Linear Regression

$$f(w) = \sum_{i=1}^{n} \left(-\left(x_i - \overline{x} \right) w + \left(y_i - \overline{y} \right) \right)^2$$

$$\frac{df}{dw} = \sum_{i=1}^{n} 2 \left(-\left(x_i - \overline{x} \right) w + \left(y_i - \overline{y} \right) \right) \left(-\left(x_i - \overline{x} \right) \right)$$

$$\frac{d^2 f}{dw^2} = \sum_{i=1}^{n} 2 \left(x_i - \overline{x} \right)^2$$

Minima: Scaled Variance & Covariance

$$\frac{df}{dw} = 0$$

$$\sum_{i=1}^{n} 2\left(-\left(x_i - \bar{x}\right)w + \left(y_i - \bar{y}\right)\right)\left(-\left(x_i - \bar{x}\right)\right) = 0$$

$$\sum_{i=1}^{n}\left(\left(x_i - \bar{x}\right)^2 w - \left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)\right) = 0$$

$$\sum_{i=1}^{n}\left(\left(x_i - \bar{x}\right)^2 w\right) - \sum_{i=1}^{n}\left(\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)\right) = 0$$

Minima:

Sum of Squared Differences & Sum of the Product of Differences

$$\sum_{i=1}^{n}\left(\left(x_i-\bar{x}\right)^2 w\right)-\sum_{i=1}^{n}\left(\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)\right)=0$$

$$w\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2=\sum_{i=1}^{n}\left(\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)\right)$$

$$w=\frac{\sum_{i=1}^{n}\left(\left(x_i-\bar{x}\right)\left(y_i-\bar{y}\right)\right)}{\sum_{i=1}^{n}\left(x_i-\bar{x}\right)^2}$$

Linear Regression

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n} \qquad \bar{y} = \frac{\sum_{i=1}^{n} y_i}{n} \qquad w = \frac{\sum_{i=1}^{n}\left(\left(x_i - \bar{x}\right)\left(y_i - \bar{y}\right)\right)}{\sum_{i=1}^{n}\left(x_i - \bar{x}\right)^2}$$

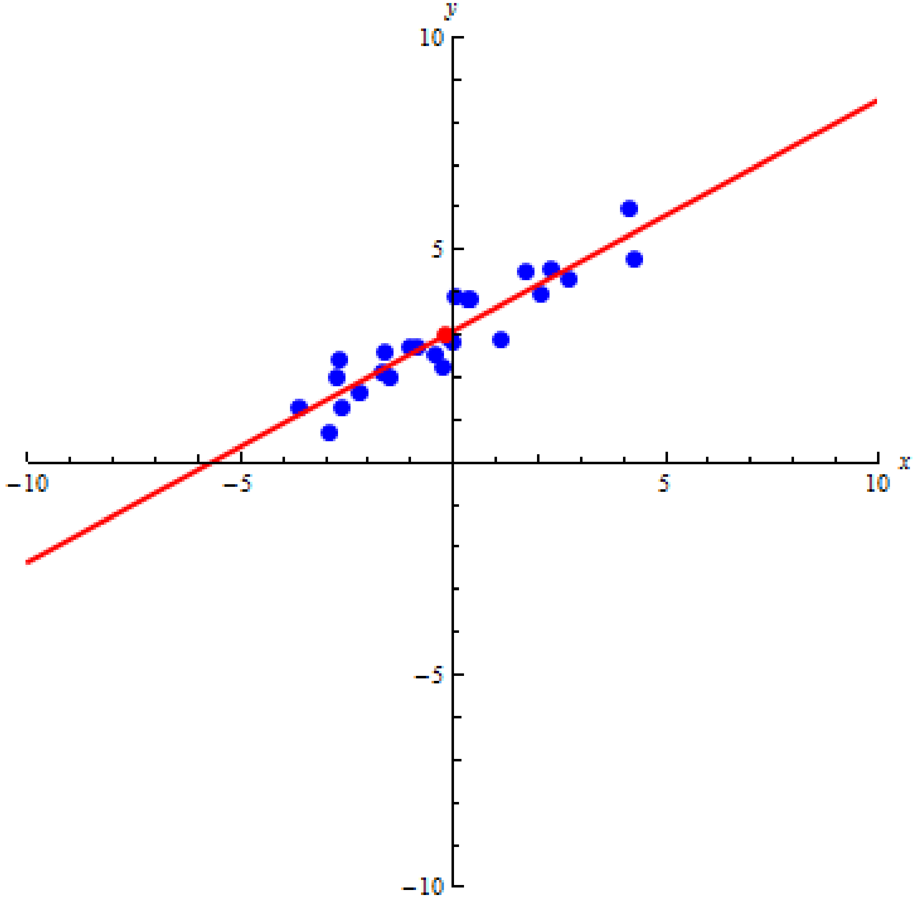
$$y = w\left(x - \bar{x}\right) + \bar{y}$$

$$y = \underbrace{\left(w\right)}_{m} x + \underbrace{\left(-w\bar{x} + \bar{y}\right)}_{b}$$

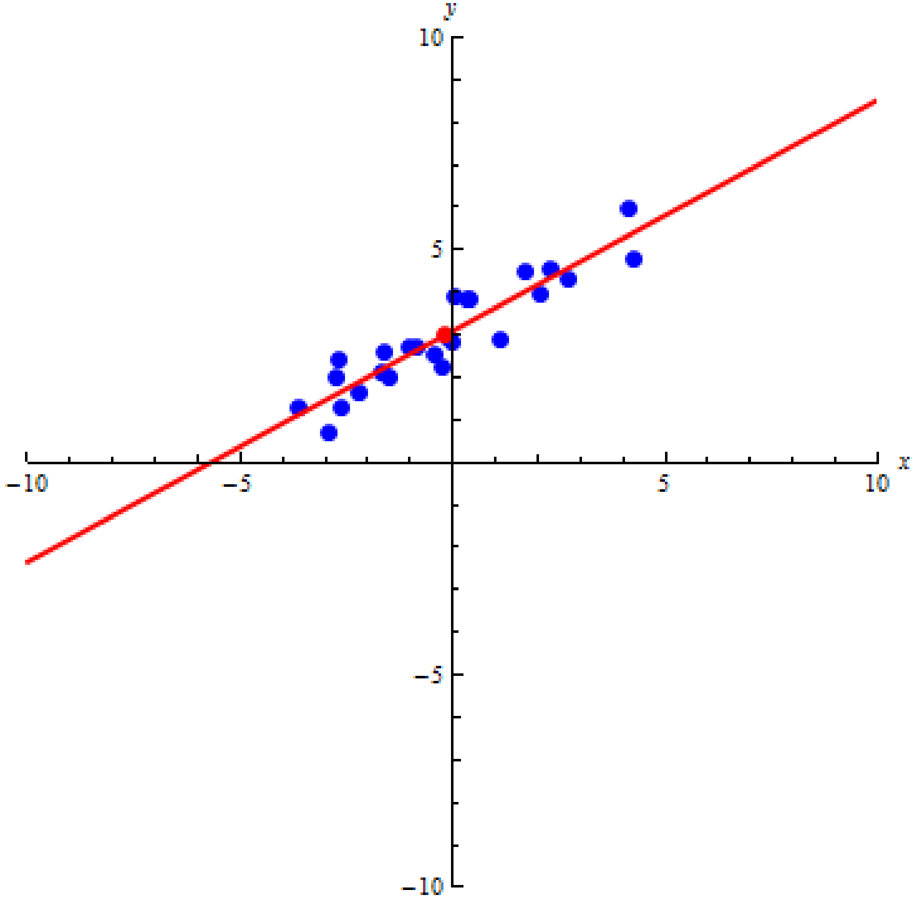
Linear Regression

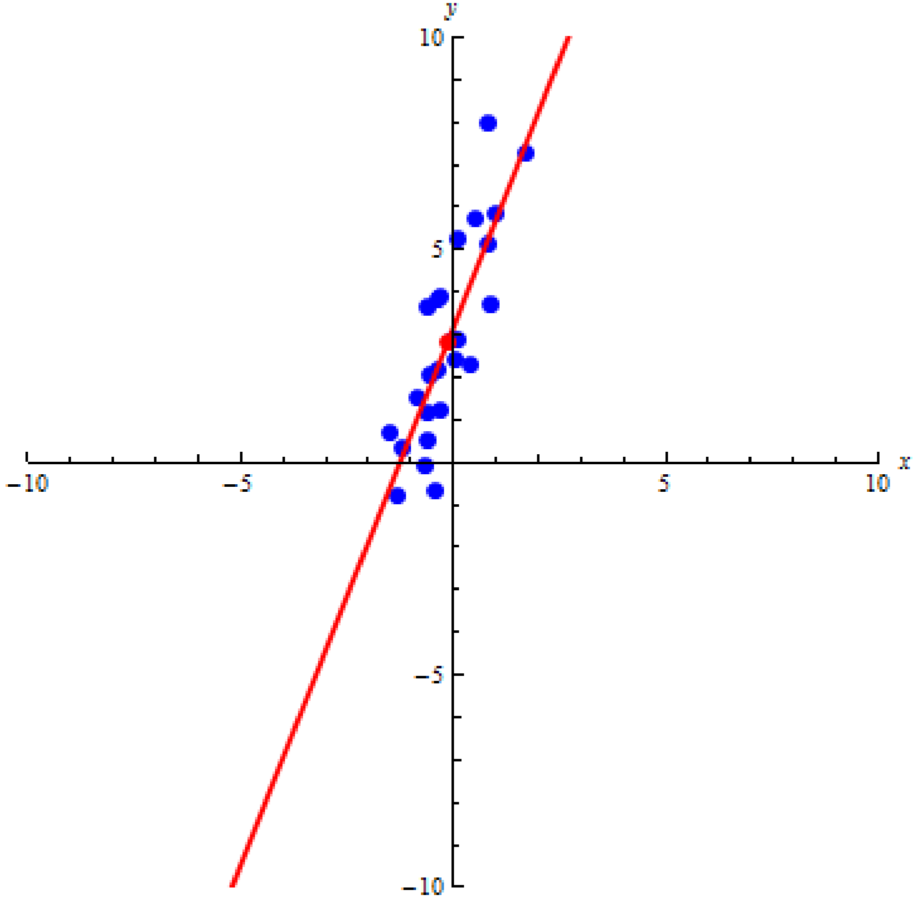
- Slope: `0.543745`
- Mean: `{-0.208094,2.98517}`
- Equation:

$$y = 0.543745x + 3.09832$$

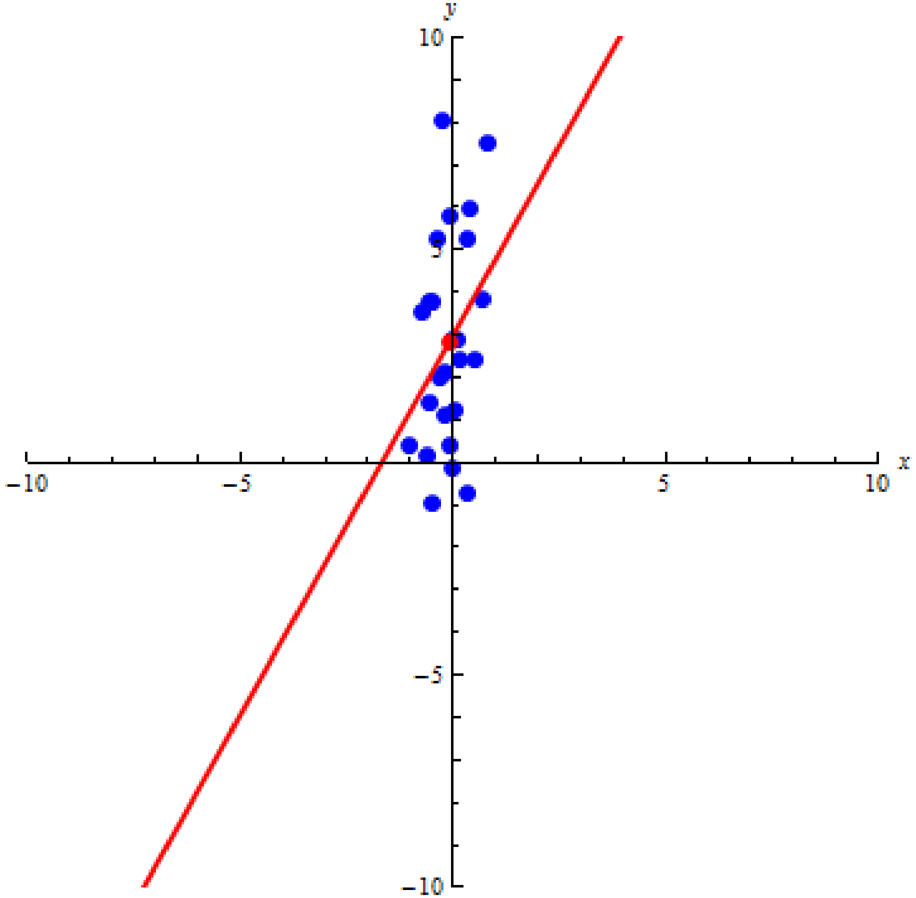


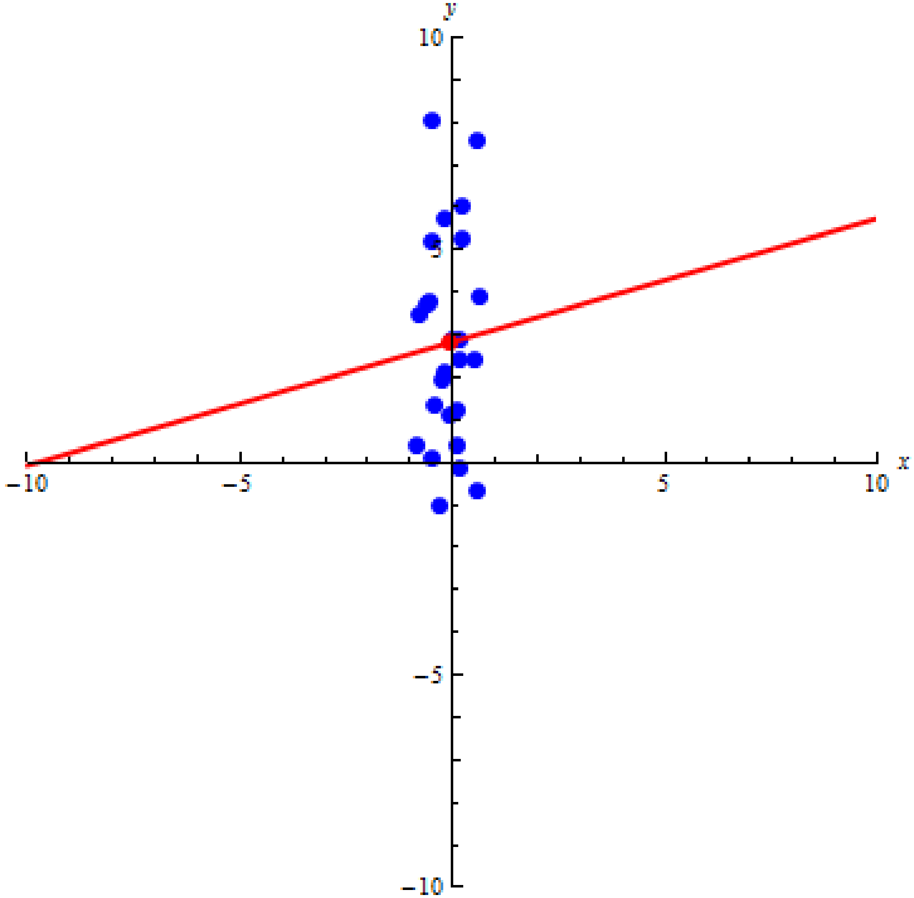
Independent Variable Bias





Independent Variable Bias





Online Mean

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

$$\bar{x} = \frac{\sum_{i=1}^{n-1} x_i + x_n}{n} \qquad \bar{x} = \frac{\sum_{i=1}^{n-1} x_i}{n} \cdot \frac{n-1}{n-1} + \frac{x_n}{n} \qquad \bar{x}_n = \bar{x}_{n-1} \cdot \frac{n-1}{n} + \frac{x_n}{n}$$

$$\bar{x} = \frac{\sum_{i=1}^{n-1} x_i}{n} + \frac{x_n}{n} \qquad \bar{x} = \frac{\sum_{i=1}^{n-1} x_i}{n-1} \cdot \frac{n-1}{n} + \frac{x_n}{n} \qquad \bar{x}_n = \frac{\bar{x}_{n-1}\left(n-1\right) + x_n}{n}$$

One Way To Program Simple Linear Regression

```
/*
Filename: main.cpp
To compile and run on linprog4.cs.fsu.edu: g++47 -o main.exe main.cpp -std=c++11 -O3 -Wall -Wextra -Werror -static && ./main.exe
*/
#include <iostream>

template <size_t n> std::ostream & operator << ( std::ostream & o, char const ( & c )[n] ) { for ( size_t i = 0; i < n - 1; ++i ) std::cout << c[ i ]; return o; }
template <typename T, size_t n > std::ostream & operator << ( std::ostream & o, T const ( & v )[n] ) {
		auto i = std::begin( v ); auto end = std::end( v ); o << '{'; if ( i != end ) { o << *i; for ( ++i; i != end; ++i ) o << ',' << *i; } o << '}'; return o;
}

int main () {
		using number = double;
		enum dimensions { X, Y, DIMENSIONS };
		number points[][ DIMENSIONS ]{
				{-2.70238,2.43155}, {-2.18612,1.63634}, {0.405141,3.84976},{-1.03072,2.7144}, {1.10033,2.89639},
				{-1.63906,2.56916}, {2.27983,4.57127}, {-0.836348,2.70824}, {-2.90988,0.685828}, {-0.104817,2.95222},
				{-0.226538,2.24849}, {-2.64364,1.28981}, {-0.00953108,2.84022}, {0.336282,3.86626}, {4.12633,5.94993},
				{1.70053,4.51368}, {-1.4793,1.99986}, {0.0467884,3.91811}, {-1.70285,2.12758}, {-3.61035,1.26436},
				{2.08504,3.94459}, {4.23512,4.80965}, {2.70993,4.30984}, {-2.72741,1.97363}, {-0.418723,2.55797}
		};
		number mean[ DIMENSIONS ]{ 0 };
		size_t n{ 1 };
		for ( auto & point : points ) {
				mean[ X ] = ( mean[ X ] * ( n - 1 ) + point[ X ] ) / n;
				mean[ Y ] = ( mean[ Y ] * ( n - 1 ) + point[ Y ] ) / n;
				++n;
		}
		std::cout << "mean: " << mean << '\n';
		number residuals[ DIMENSIONS ]{ 0 };
		number sum_of_product_of_differences_xy{ 0 };
		number sum_of_squared_differences_xx{ 0 };
		for ( auto & point : points ) {
				residuals[ X ] = point[ X ] - mean[ X ];
				residuals[ Y ] = point[ Y ] - mean[ Y ];
				sum_of_product_of_differences_xy += residuals[ X ] * residuals[ Y ];
				sum_of_squared_differences_xx    += residuals[ X ] * residuals[ X ];
		}
		number w{ sum_of_product_of_differences_xy / sum_of_squared_differences_xx };
		std::cout << "w: " << w << '\n';
		number m{ w };
		number b{ - w * mean[ X ] + mean[ Y ] };
		std::cout << "y = " << m << " * x + " << b << '\n';
}
```

Conclusions

- The sum of parabolas is a parabola.
- Parabolas produce a global minima.
- Parabolas can be used to produce Means and Linear Regressions from Variances and Covariances.
- As presented, Linear Regression is biased toward the independent variable; moreover, the type of Linear Regression cannot produce a line that is straight up and down regardless of the data.
- The Online Mean can be used to calculate the mean of an unknown amount of data or the mean of continuously generated data.