Introduction to the Mathematics of Classification Part 1

by

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for

Introduction to Artificial Intelligence
(CAP 4601)

Agenda

- If Statement
- Equation of a Plane
- Level Set
- Normal Vector
- Functions
- Rotation
- Composition of Functions
- Level Set
- If Statement
- Translation

If Statement

- Given some information
  - Let's say: `double f[] = { x, y }`
- We want to classify that information
  - Let's say: `0 <= f[0]` . . . so either `true` or `false`
- Based on that classification, we want to do something
  - Let's say:

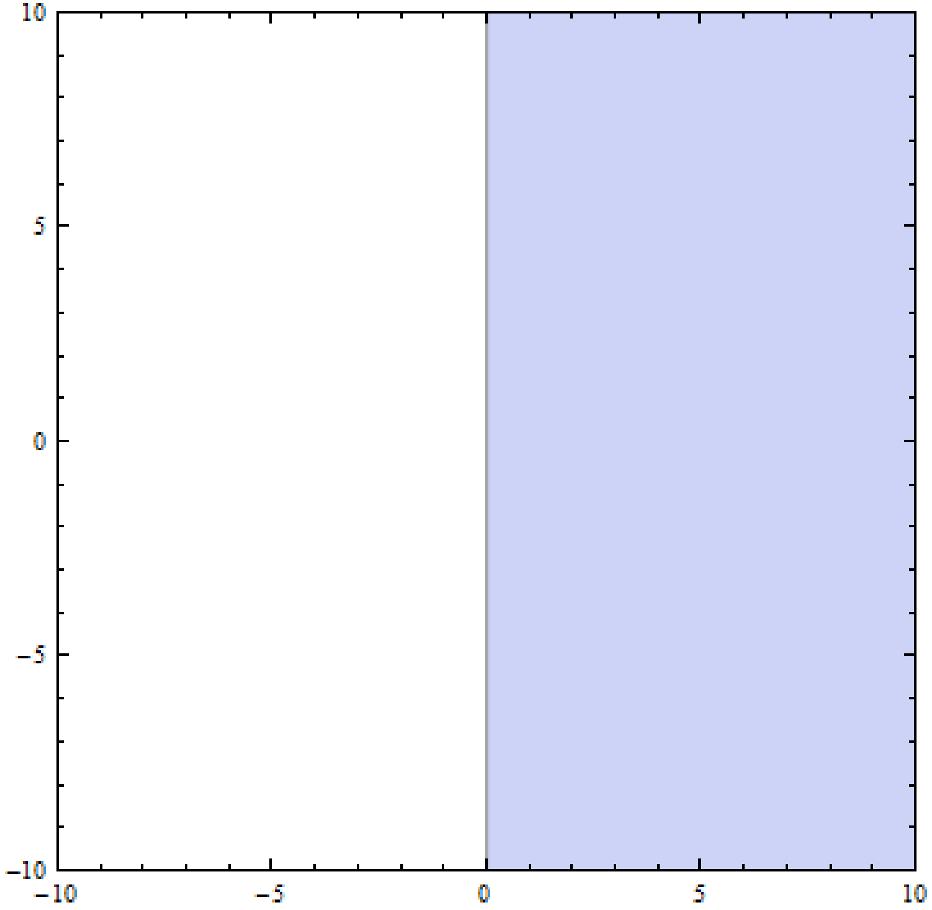
```
if ( 0 <= f[0] ) {
        // Do stuff because 0 <= f[0] is true
} else {
        // Do other stuff because 0 <= f[0] is false
}
```

If Statement

- Given some information
  - Let's say: `f = { x, y }`
- We want to classify that information
  - Let's say: `0 <= x` . . . so either `true` or `false`
- Based on that classification, we want to do something
  - Let's say:

```
if ( 0 <= x ) {
      // Do stuff because 0 <= x is true
} else {
      // Do other stuff because 0 <= x is false
}
```

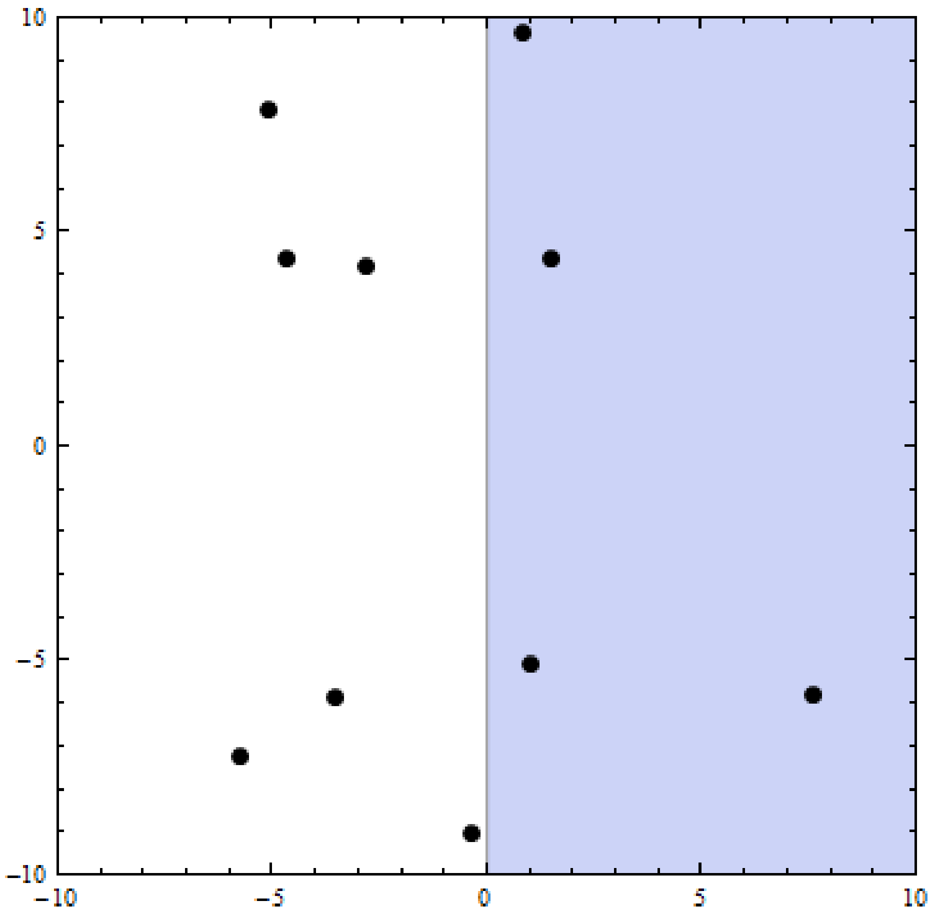
If Statement



- Given:

```
{
     {-4.68306,  4.355  },
     { 1.52441,  4.36277},
     { 7.59389, -5.7927 },
     {-2.79211,  4.16882},
     { 0.834487, 9.66578},
     {-3.53218, -5.89357},
     {-0.368558,-9.04035},
     {-5.7306,  -7.24471},
     {1.00427,  -5.10584},
     {-5.10852,  7.87084}
}
```

If Statement



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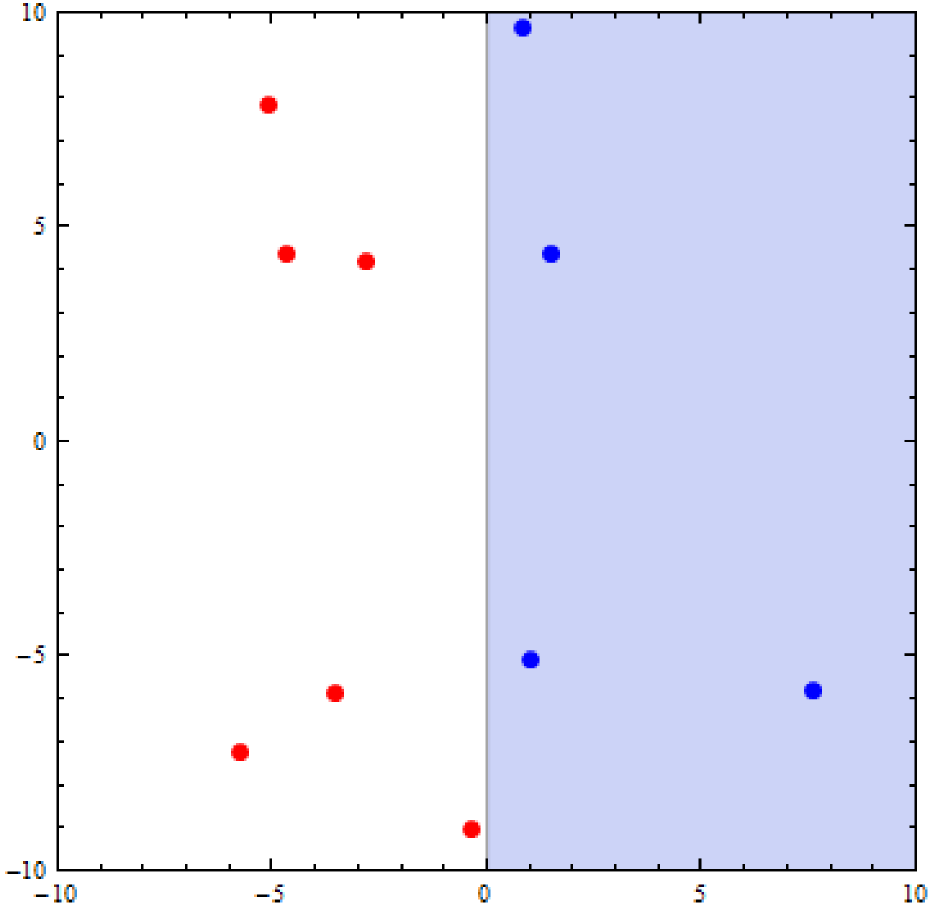
If Statement



- **Given:**

```
{
    {-4.68306,   4.355   },
    { 1.52441,   4.36277},
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    {-5.7306,   -7.24471},
    {1.00427,   -5.10584},
    {-5.10852,   7.87084}
}
```

Equation of a Plane

$$f(\mathbf{x}) = \vec{\mathbf{n}} \bullet (\mathbf{x} - \mathbf{x}_0)$$

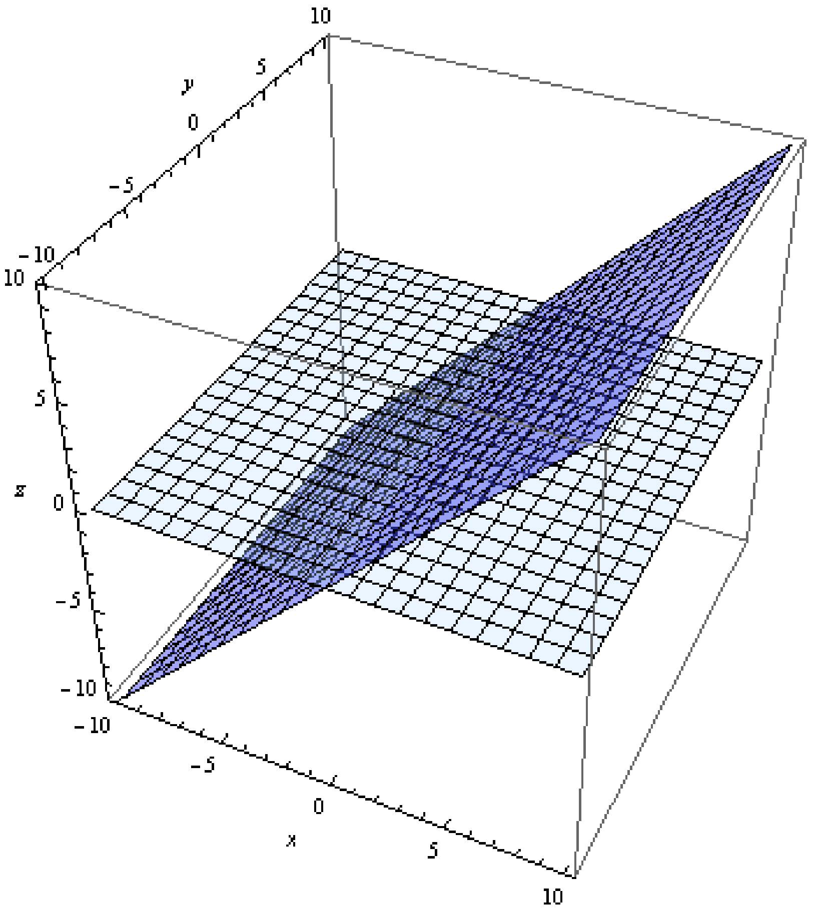
where

$\vec{\mathbf{n}}$ is the normal vector,

$\mathbf{x}$ is the point of independent variables, and

$\mathbf{x}_0$ is a point of constant values.

Equation of a Plane



$$f(\mathbf{x}) = \vec{\mathbf{n}} \bullet (\mathbf{x} - \mathbf{x}_0)$$

$$f(x, y) = (1, 0) \bullet ((x, y) - (0, 0))$$

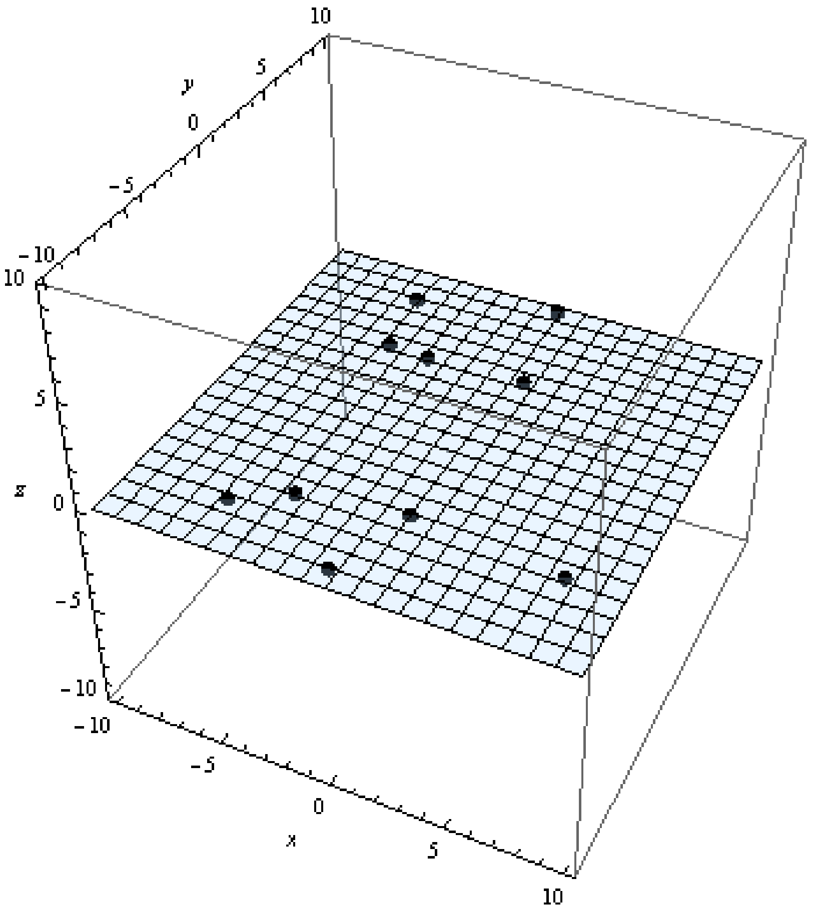
$$f(x, y) = (1, 0) \bullet (x - 0, y - 0)$$

$$f(x, y) = (1, 0) \bullet (x, y)$$

$$f(x, y) = 1 \cdot x + 0 \cdot y$$

$$f(x, y) = x$$

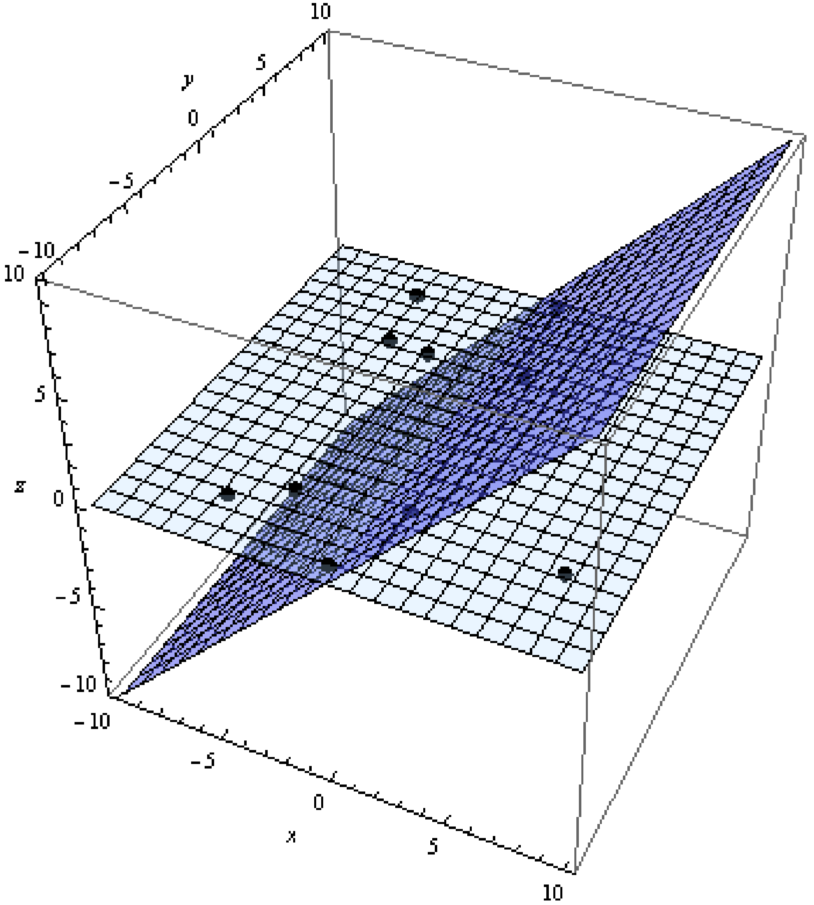
Level Set



$$f\left(x, y\right) = c$$
where $c \in \mathbb{R}$

For classification, we can classify points $\left(x, y\right)$ based on the level set such that $f\left(x, y\right) < c$ and $f\left(x, y\right) \geq c$

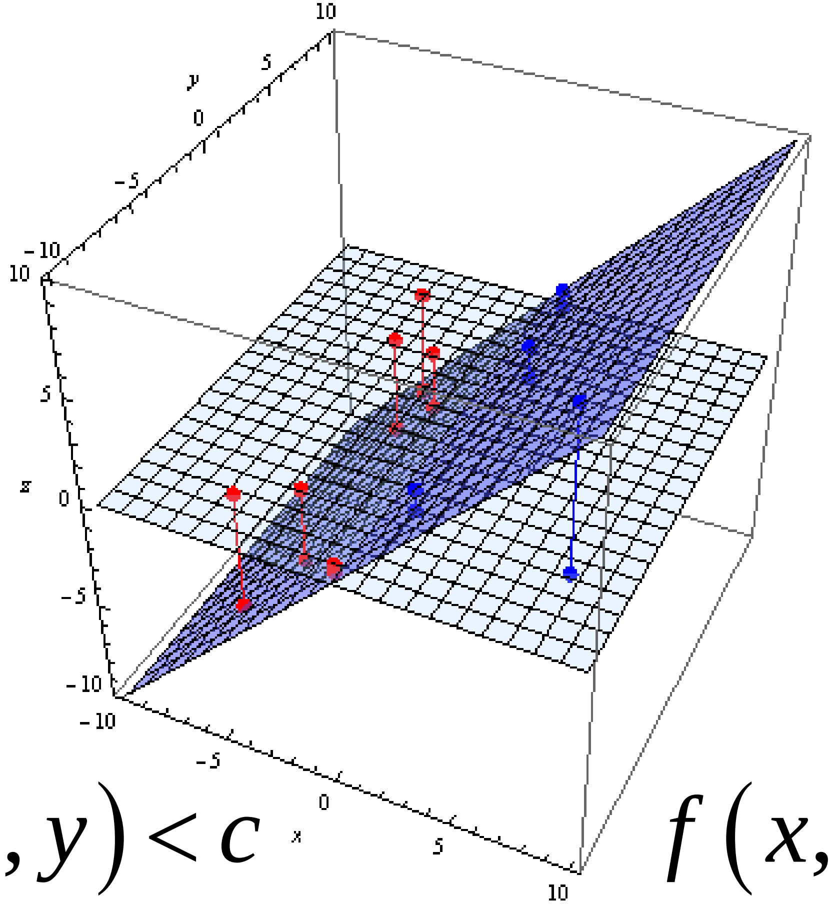
Level Set



$$f(\mathbf{x}) = \vec{\mathbf{n}} \bullet (\mathbf{x} - \mathbf{x}_0)$$

$$f(x, y) = x$$

Level Set



$$f\left(x,y\right)<c$$

$$x<0$$

$$f\left(x,y\right)\geq c$$

$$x\geq 0$$

$$f\left(\mathbf{x}\right)=\vec{\mathbf{n}}\bullet\left(\mathbf{x}-\mathbf{x}_0\right)$$

$$f\left(x,y\right)=x$$

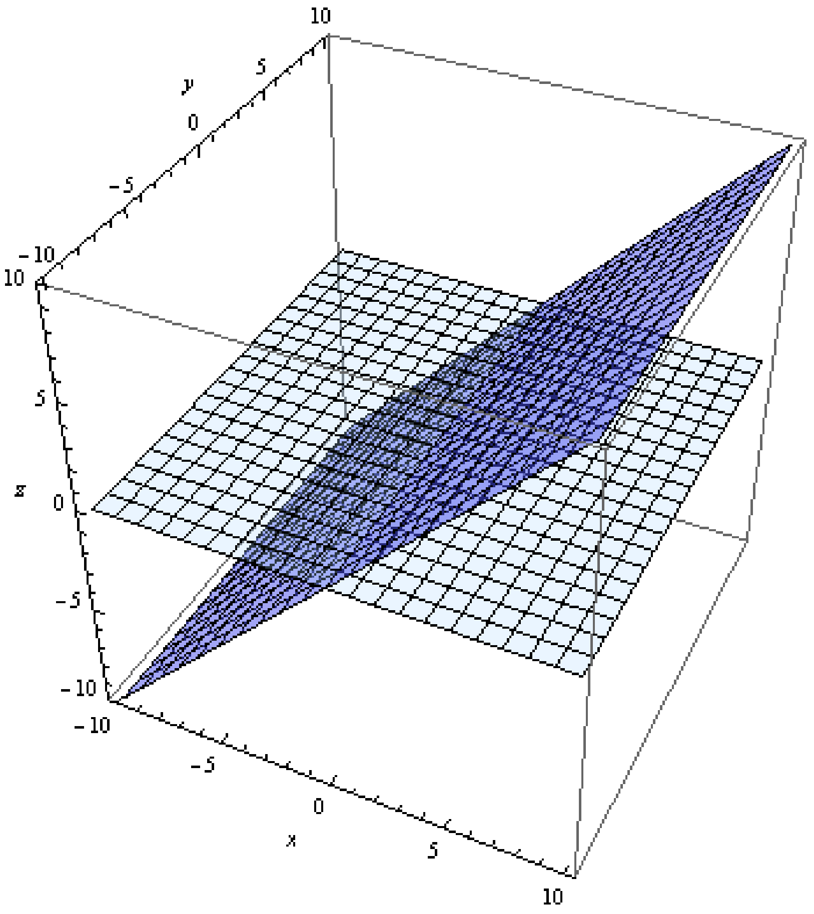
Level Set:

$$f\left(x,y\right)=\quad c$$

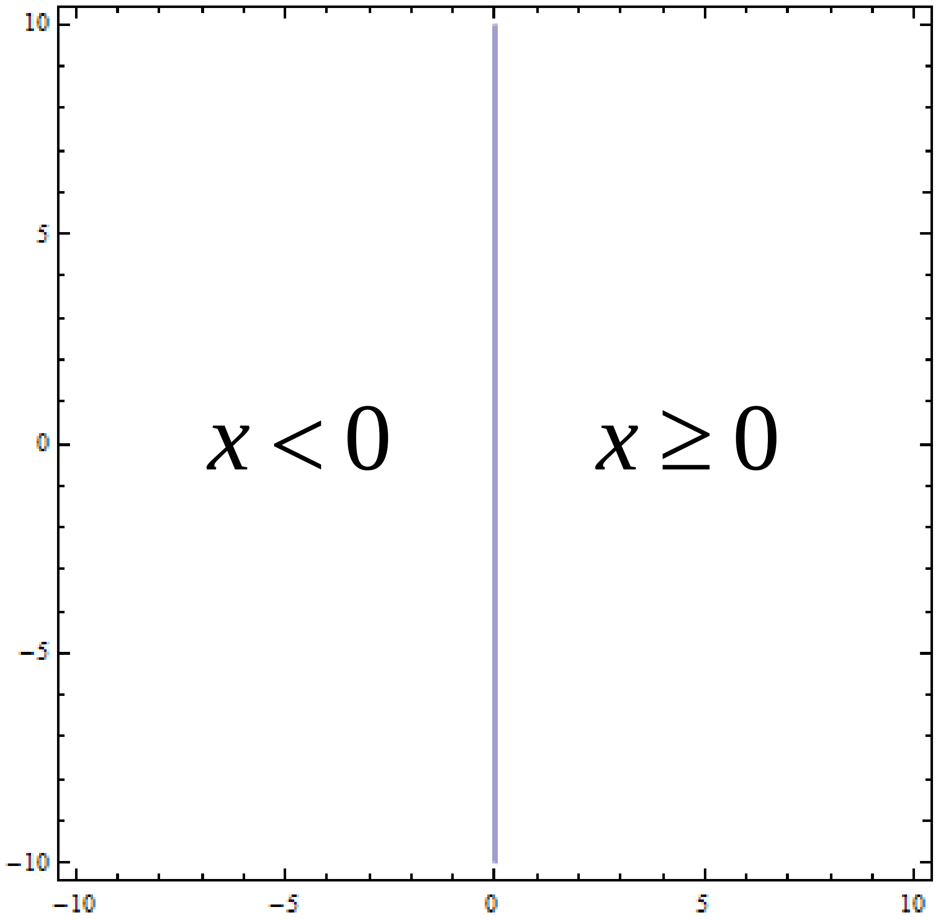
Let c=0

$$x=0$$

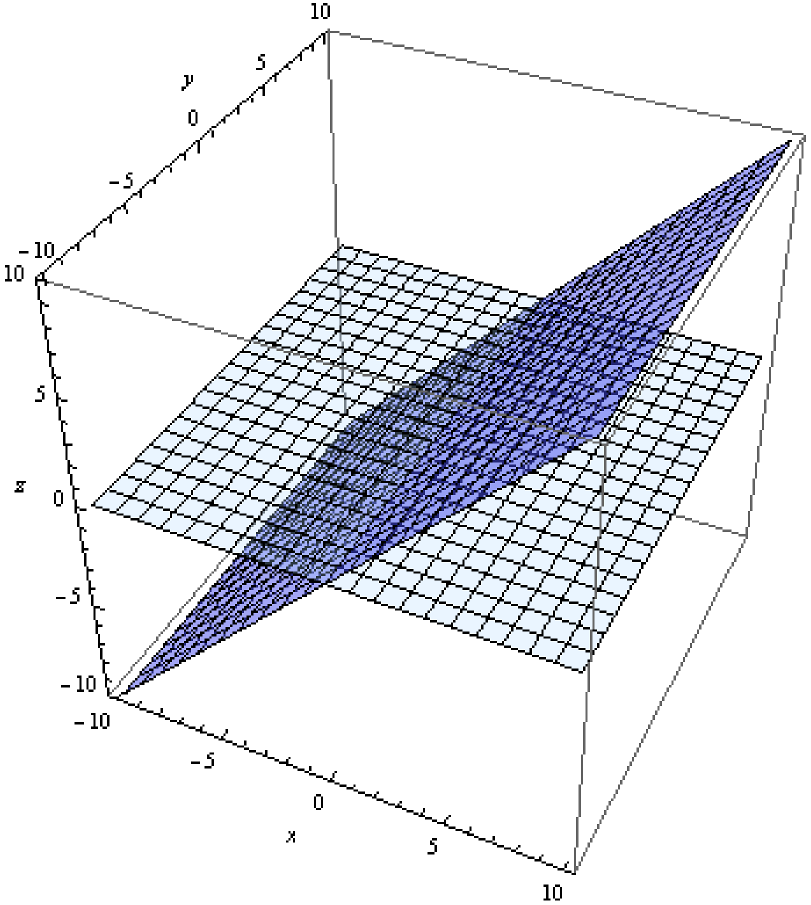
Level Set

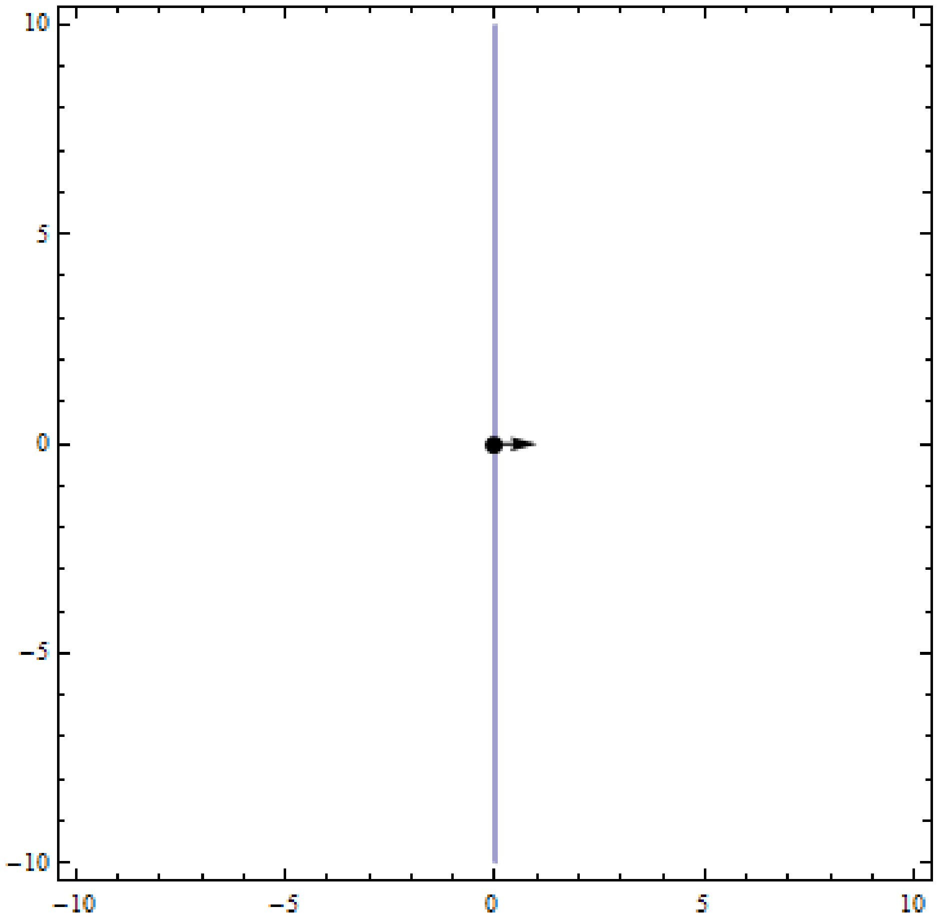


$x = 0$

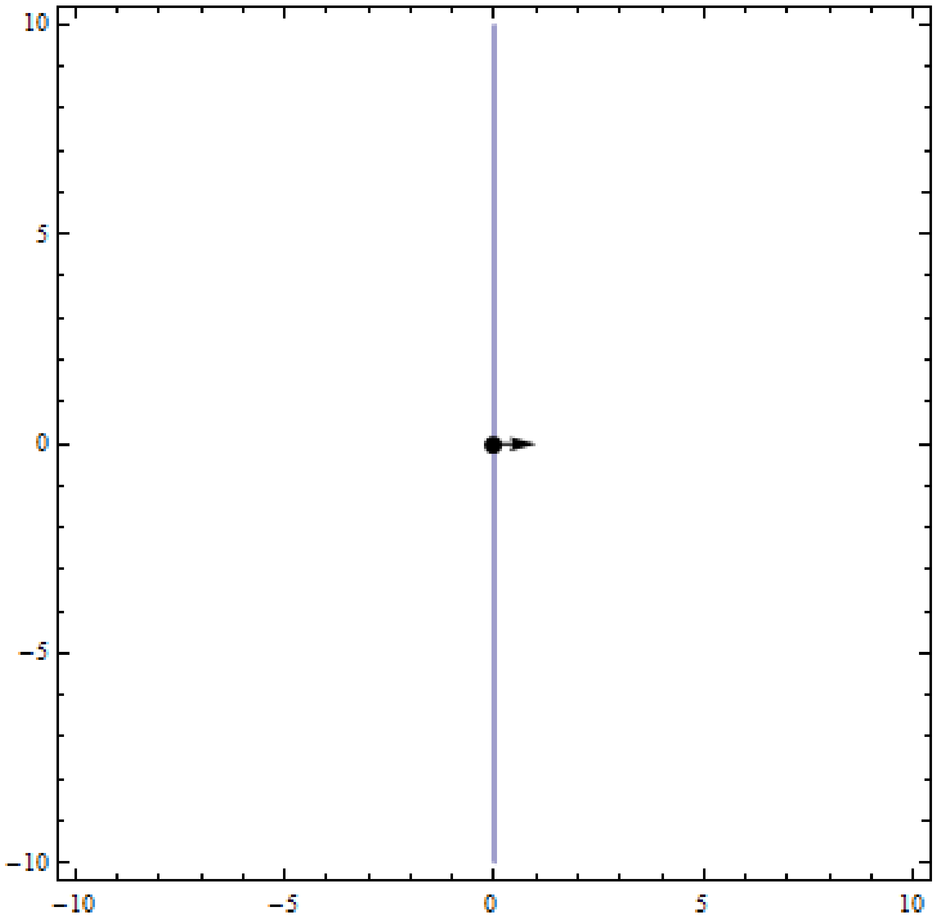


Normal Vector





Equation of a Plane: Origin



$$f(\mathbf{x}) = \vec{\mathbf{n}} \bullet (\mathbf{x} - \mathbf{x}_0)$$

$$f(x,y) = (1,0) \bullet ((x,y) - (0,0))$$

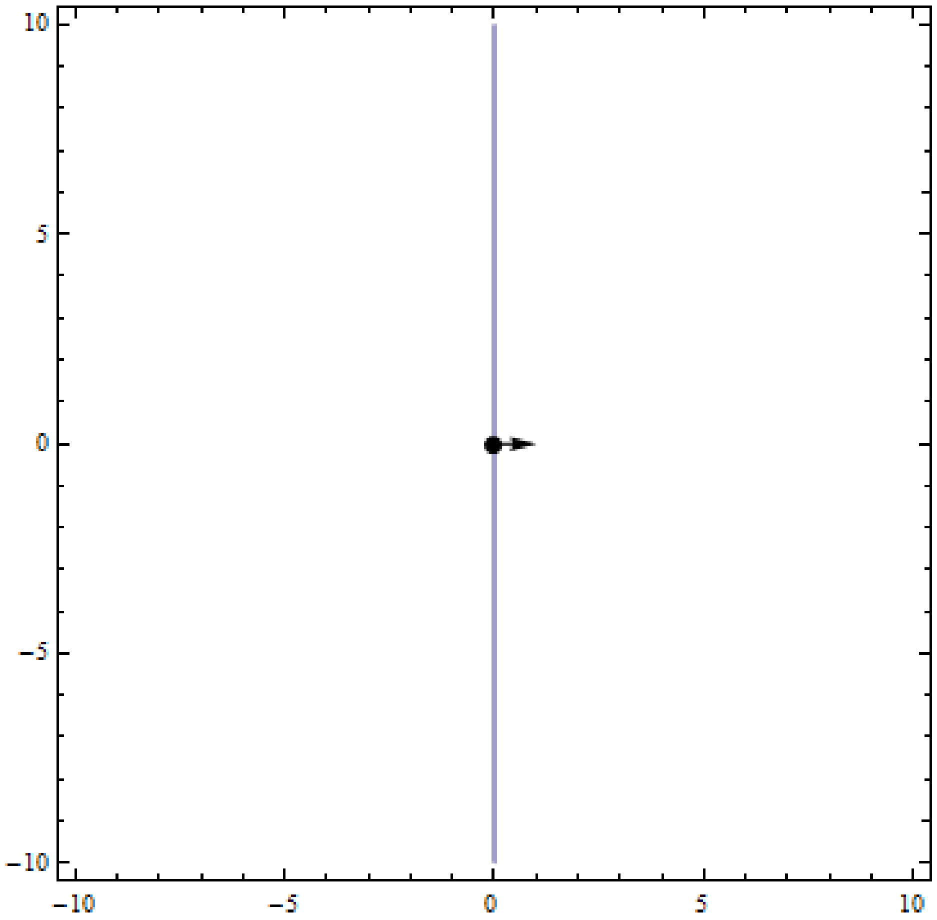
$$f(x,y) = (1,0) \bullet (x - 0, y - 0)$$

$$f(x,y) = (1,0) \bullet (x,y)$$

$$f(x,y) = 1 \cdot x + 0 \cdot y$$

$$f(x,y) = x$$

Equation of a Plane: Normal Vector



$$f(\mathbf{x}) = \vec{\mathbf{n}} \bullet (\mathbf{x} - \mathbf{x}_0)$$

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$$f(x, y) = (1, 0) \bullet (x - 0, y - 0)$$

$$f(x, y) = (1, 0) \bullet (x, y)$$

$$f(x, y) = 1 \cdot x + 0 \cdot y$$

$$f(x, y) = x$$

Normal Vector

- Why the (1,0)? Why not (5,0)? Or something else?
  - Because it's "normalized" (i.e. its magnitude is equal to one) and that property has desirable qualities that we'll discuss later. For now, here is the definition of normalized:

$$\vec{\mathbf{n}}_{\text{normalized}} = \frac{\vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|} = \frac{(n_1, n_2, \ldots, n_m)}{\|\vec{\mathbf{n}}\|} = \left(\frac{n_1}{\|\vec{\mathbf{n}}\|}, \frac{n_2}{\|\vec{\mathbf{n}}\|}, \ldots, \frac{n_m}{\|\vec{\mathbf{n}}\|} \right)$$

$$\|\vec{\mathbf{n}}\| = \|(n_1, n_2, \ldots, n_m)\| = \sqrt{n_1^2 + n_2^2 + \cdots + n_m^2}$$

Normal Vector: Normalized in 2D

Let $\vec{\mathbf{n}} = \left(\cos(\theta), \sin(\theta)\right)$, then

$$\frac{\vec{\mathbf{n}}}{\|\vec{\mathbf{n}}\|} = \frac{\left(\cos(\theta), \sin(\theta)\right)}{\sqrt{\left(\cos(\theta)\right)^2 + \left(\sin(\theta)\right)^2}}$$

$$= \frac{\left(\cos(\theta), \sin(\theta)\right)}{\sqrt{1}}$$

$$= \left(\cos(\theta), \sin(\theta)\right)$$

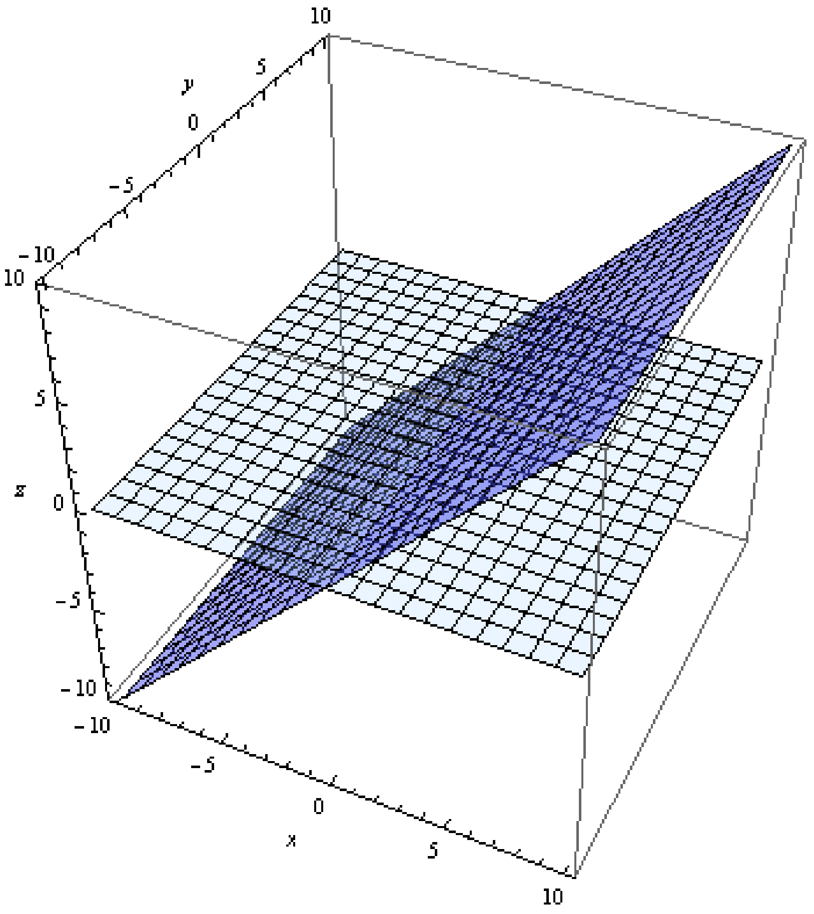
So, $\vec{\mathbf{n}}_{\text{normalized}} = \left(\cos(\theta), \sin(\theta)\right)$

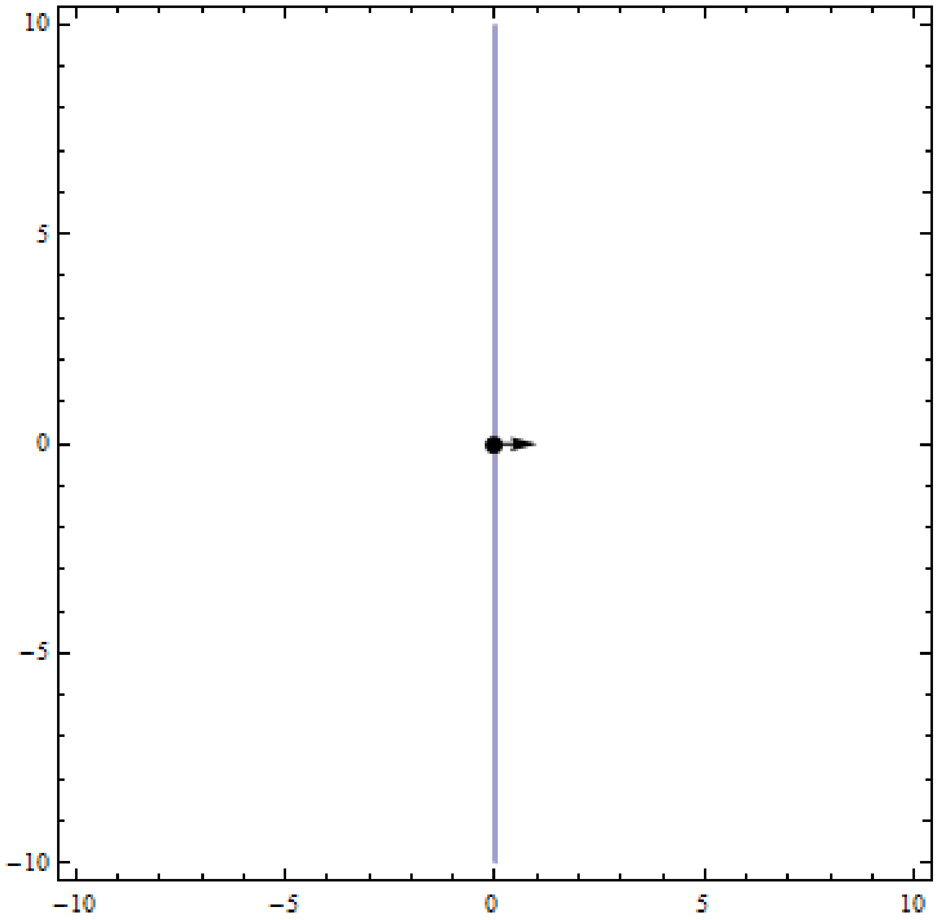
If $\theta = 0$, then

$$\vec{\mathbf{n}}_{\text{normalized}} = \left(\cos(0), \sin(0)\right)$$

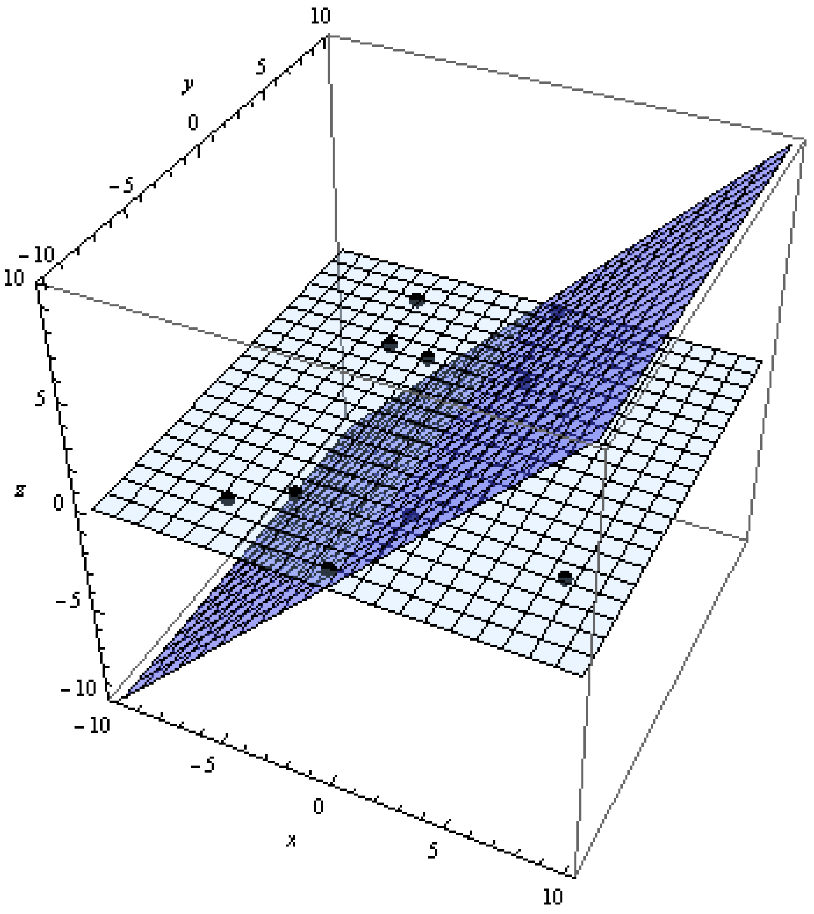
$$= (1, 0)$$

Normal Vector

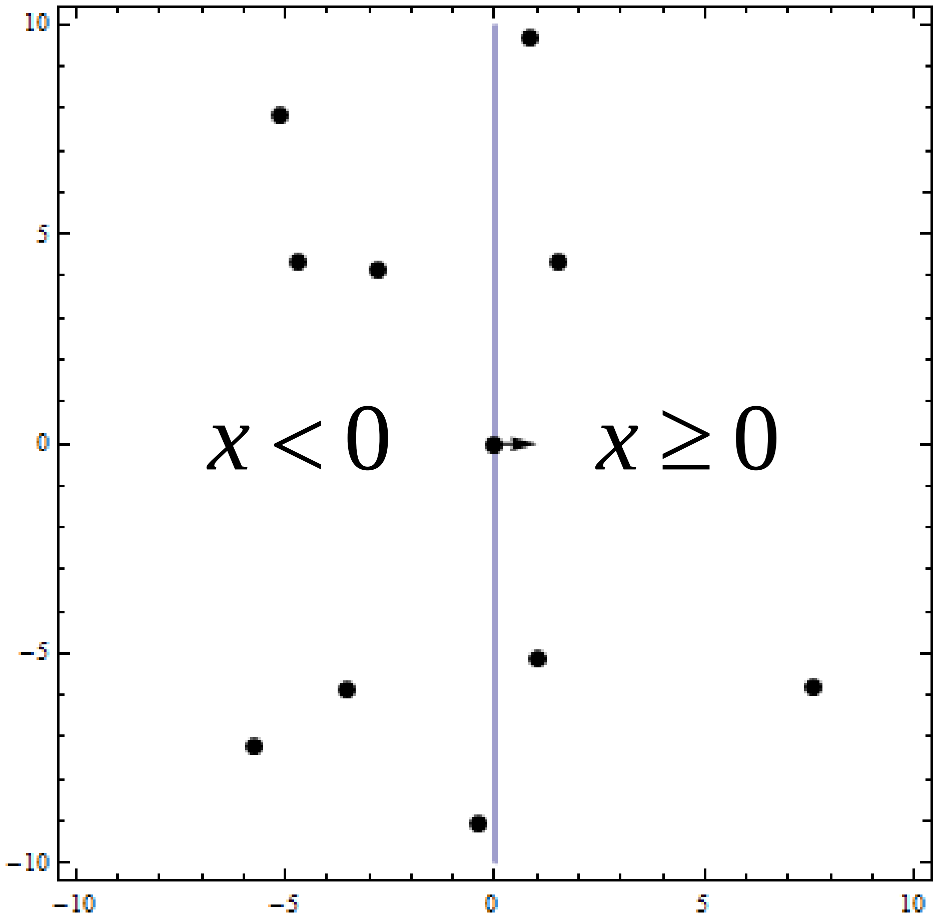




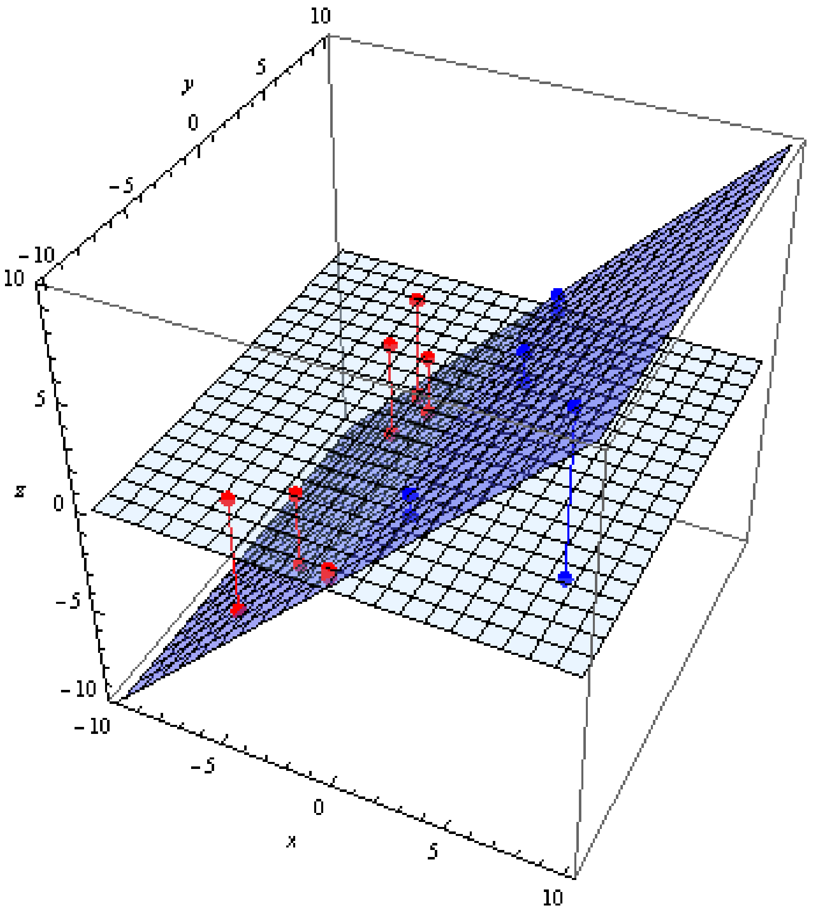
Normal Vector



$x = 0$



Normal Vector



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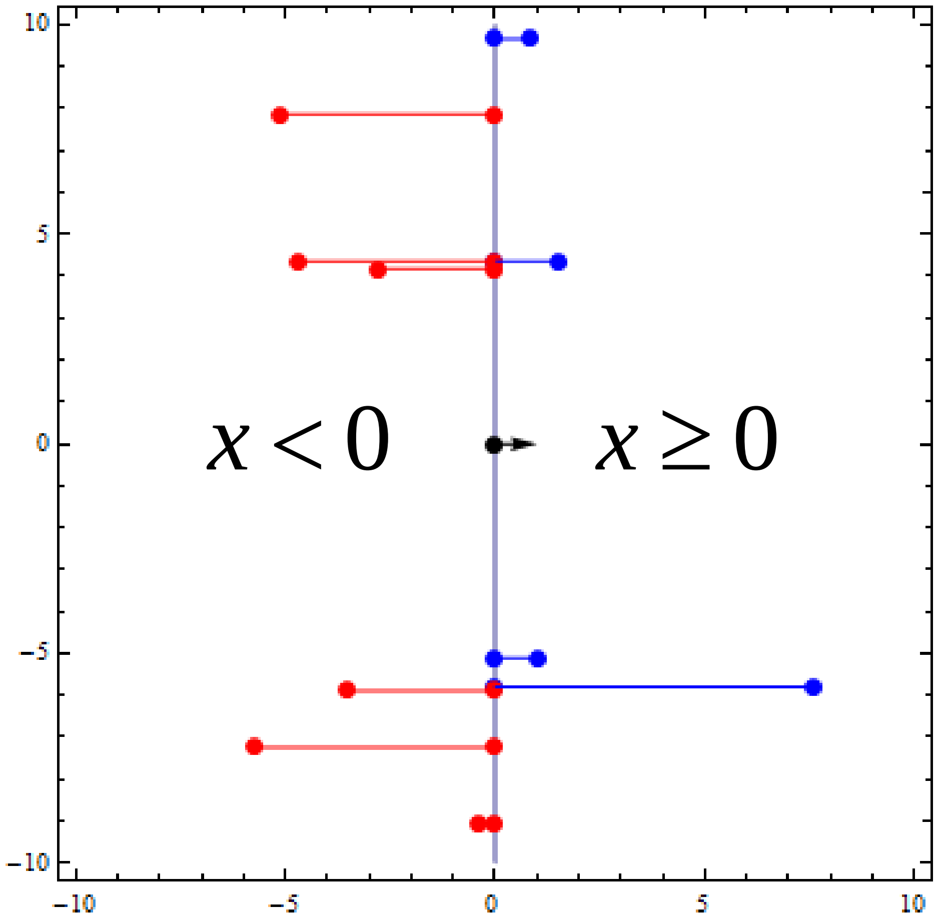


Normal Vector

- For instance:

$$\text{Given } \vec{\mathbf{n}}_{\text{normalized}} = (1,0) \text{ and } \mathbf{x}_0 = (0,0)\text{, then}$$

$$\text{if we receive } \mathbf{x} = (-4.68306, 4.355)\text{, we have}$$

$$\vec{\mathbf{n}}_{\text{normalized}} \bullet (\mathbf{x} - \mathbf{x}_0)$$

$$(1,0) \bullet ((-4.68306, 4.355) - (0,0))$$

$$(1,0) \bullet (-4.68306 - 0, 4.355 - 0)$$

$$(1,0) \bullet (-4.68306, 4.355)$$

$$(1)(-4.68306) + (0)(4.355)$$

$$-4.68306$$

Functions

$$f(x) = x \qquad x \in \mathbb{R}^1 \qquad f(x) \in \mathbb{R}^1$$

$$f(x, y) = x \qquad (x, y) \in \mathbb{R}^2 \qquad f(x, y) \in \mathbb{R}^1$$

$$f(x_1, x_2) = x_1 \qquad (x_1, x_2) \in \mathbb{R}^2 \qquad f(x_1, x_2) \in \mathbb{R}^1$$

$$f(x_1, x_2, x_3) = x_1 \qquad (x_1, x_2, x_3) \in \mathbb{R}^3 \qquad f(x_1, x_2, x_3) \in \mathbb{R}^1$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$f(x_1, x_2, \ldots, x_n) = x_1 \qquad (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \qquad f(x_1, x_2, \ldots, x_n) \in \mathbb{R}^1$$

Functions

$$f(x) = x \qquad f : \mathbb{R}^1 \to \mathbb{R}^1$$

$$f(x, y) = x \qquad f : \mathbb{R}^2 \to \mathbb{R}^1$$

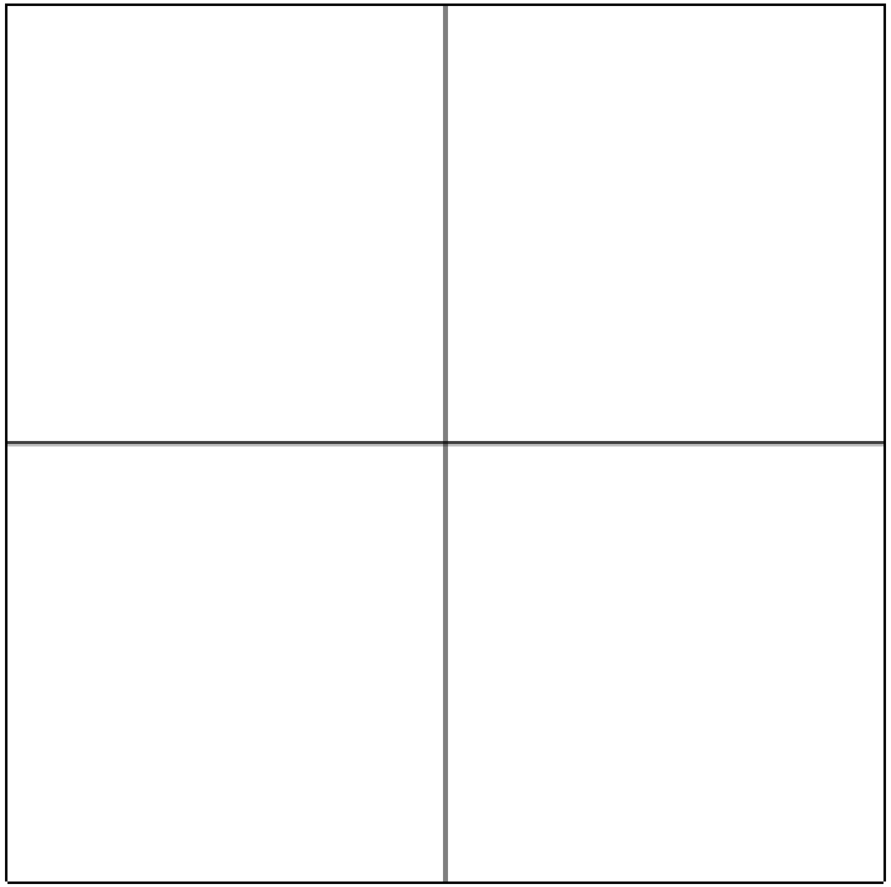
$$f(x_1, x_2) = x_1 \qquad f : \mathbb{R}^2 \to \mathbb{R}^1$$

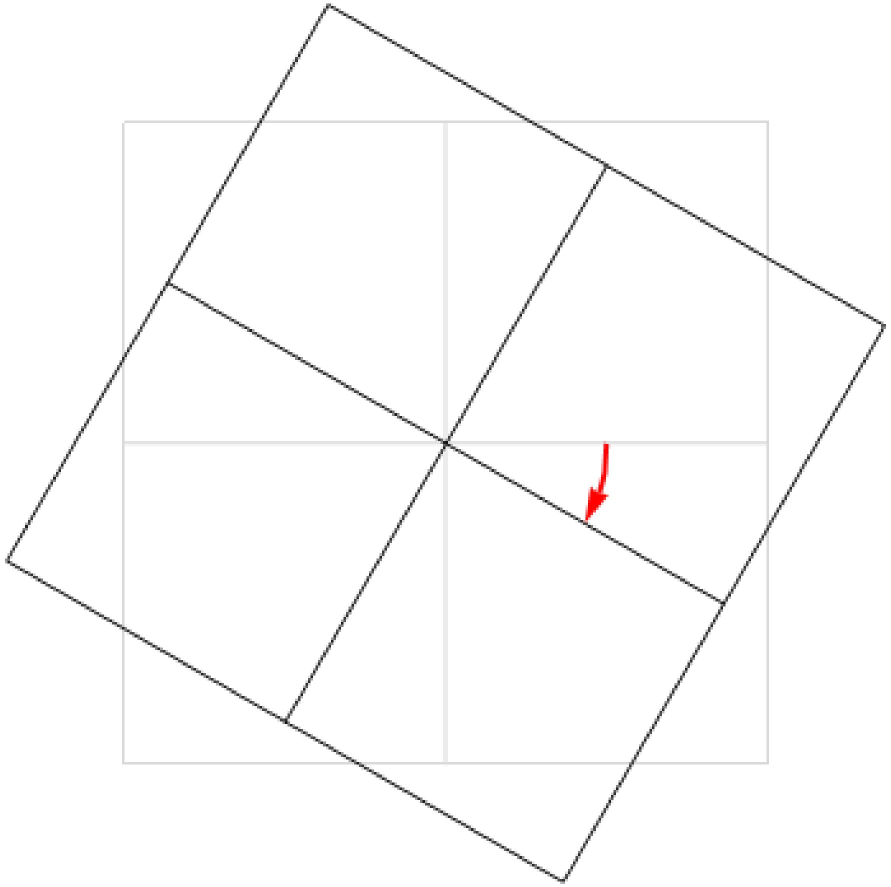
$$f(x_1, x_2, x_3) = x_1 \qquad f : \mathbb{R}^3 \to \mathbb{R}^1$$

$$\vdots \qquad \vdots$$

$$f(x_1, x_2, \ldots, x_n) = x_1 \qquad f : \mathbb{R}^n \to \mathbb{R}^1$$

Rotation





Rotation

- 2D Rotation Matrix:

$$\begin{pmatrix} \cos(\theta) & \cos\left(\theta + \frac{\pi}{2}\right) \\ \sin(\theta) & \sin\left(\theta + \frac{\pi}{2}\right) \end{pmatrix}$$

- Multiplicative Inverse of a 2D Rotation Matrix:

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ \cos\left(\theta + \frac{\pi}{2}\right) & \sin\left(\theta + \frac{\pi}{2}\right) \end{pmatrix}$$

Rotation

- 2D Rotation Matrix:

$$\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

- Multiplicative Inverse of a 2D Rotation Matrix:

$$\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

Functions

- 2D Parametric Rotation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Functions

- 2D Function Rotation:

$$g(x,y) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta)x + \sin(\theta)y \\ -\sin(\theta)x + \cos(\theta)y \end{pmatrix}$$

$$= \left(\cos(\theta)x + \sin(\theta)y, -\sin(\theta)x + \cos(\theta)y\right)$$

$$g : \mathbb{R}^2 \to \mathbb{R}^2$$

Functions: Composition of Functions

$$f\left(x,y\right)=x$$

$$g\left(x,y\right)=\left(\cos\left(\theta\right)x+\sin\left(\theta\right)y,-\sin\left(\theta\right)x+\cos\left(\theta\right)y\right)$$

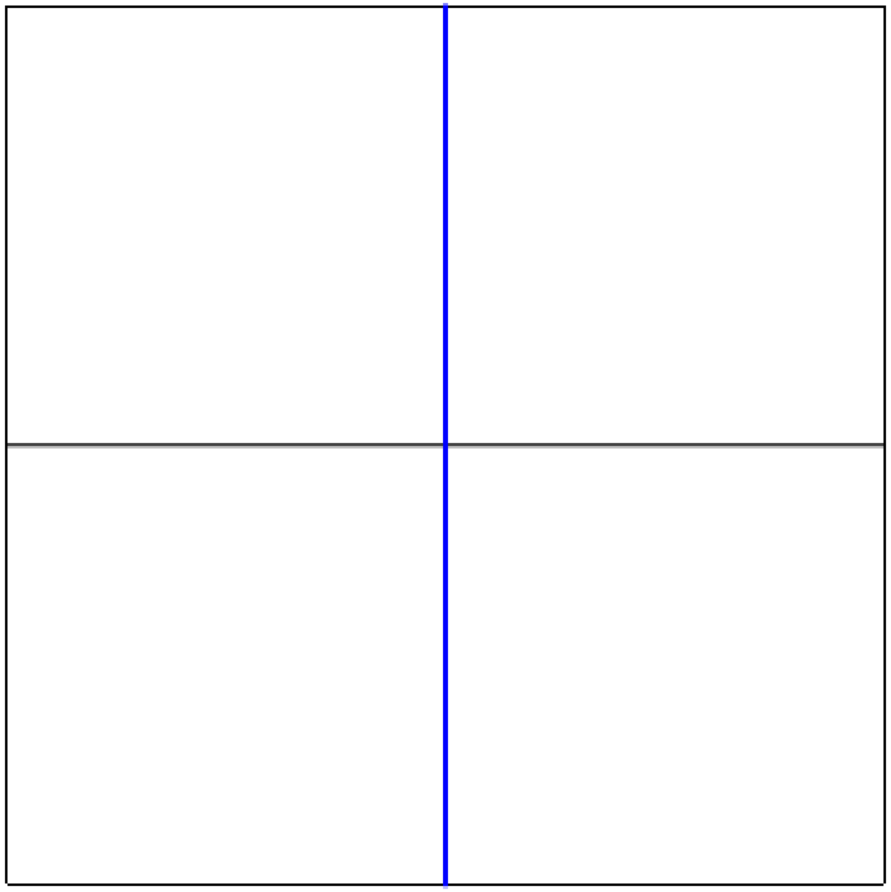
$$g:\mathbb{R}^2\to\mathbb{R}^2 \ \text{ and } \ f:\mathbb{R}^2\to\mathbb{R}^1$$

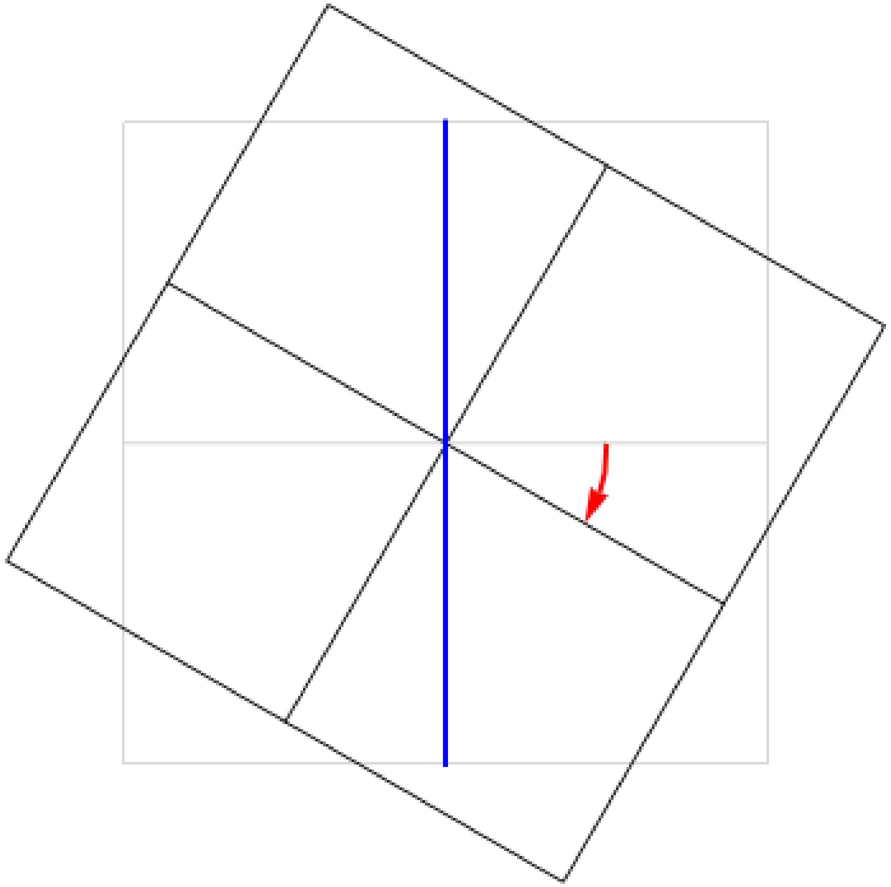
$$f\left(g\left(x,y\right)\right)=\left(\cos\left(\theta\right)x+\sin\left(\theta\right)y\right)$$

$$f\left(g\left(x,y\right)\right)=\cos\left(\theta\right)x+\sin\left(\theta\right)y \qquad f\circ g:\mathbb{R}^2\to\mathbb{R}^1$$

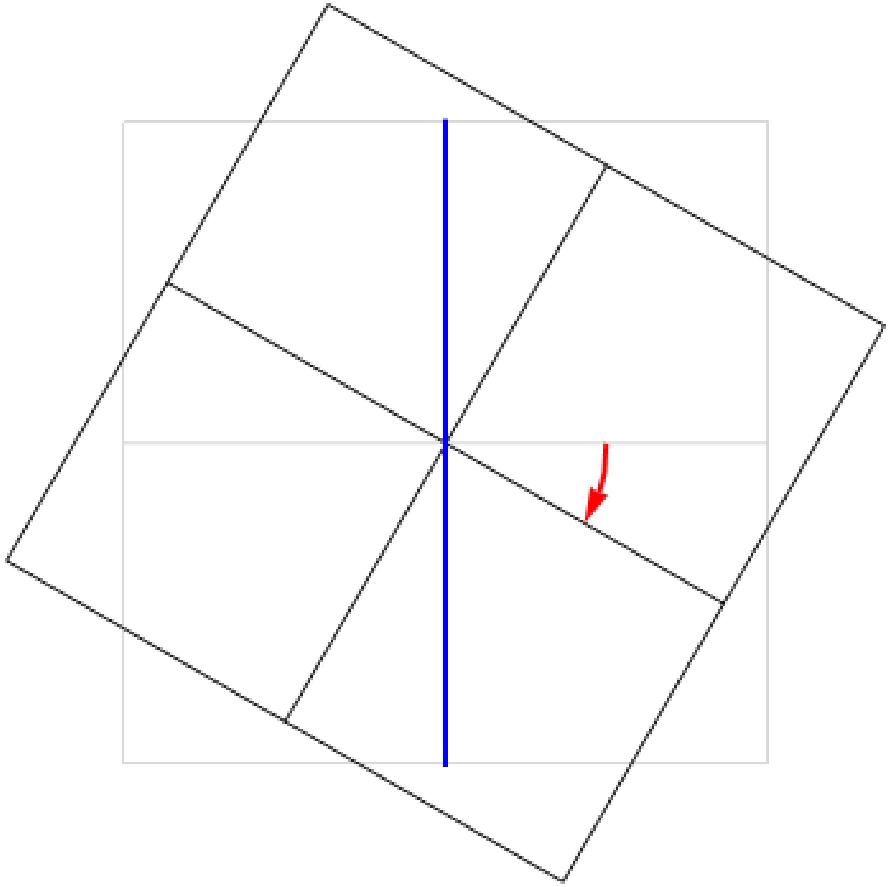
$$f\left(x,y\right)=\cos\left(\theta\right)x+\sin\left(\theta\right)y$$

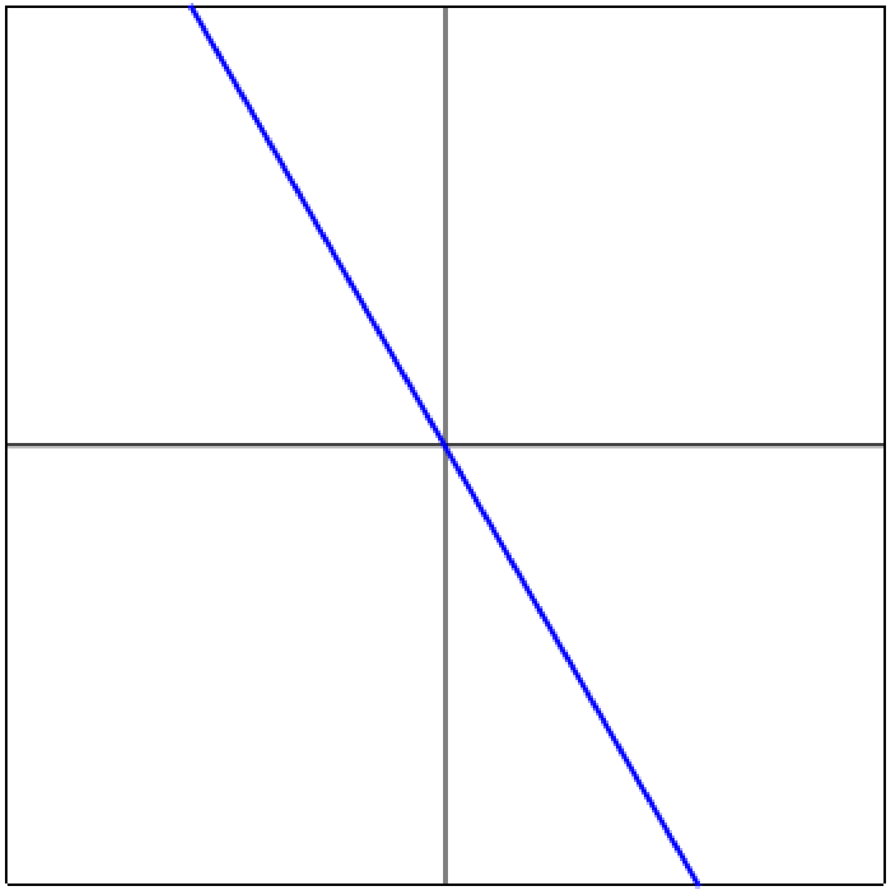
Functions: 30 Degree 2D Rotation





Functions: 30 Degree 2D Rotation





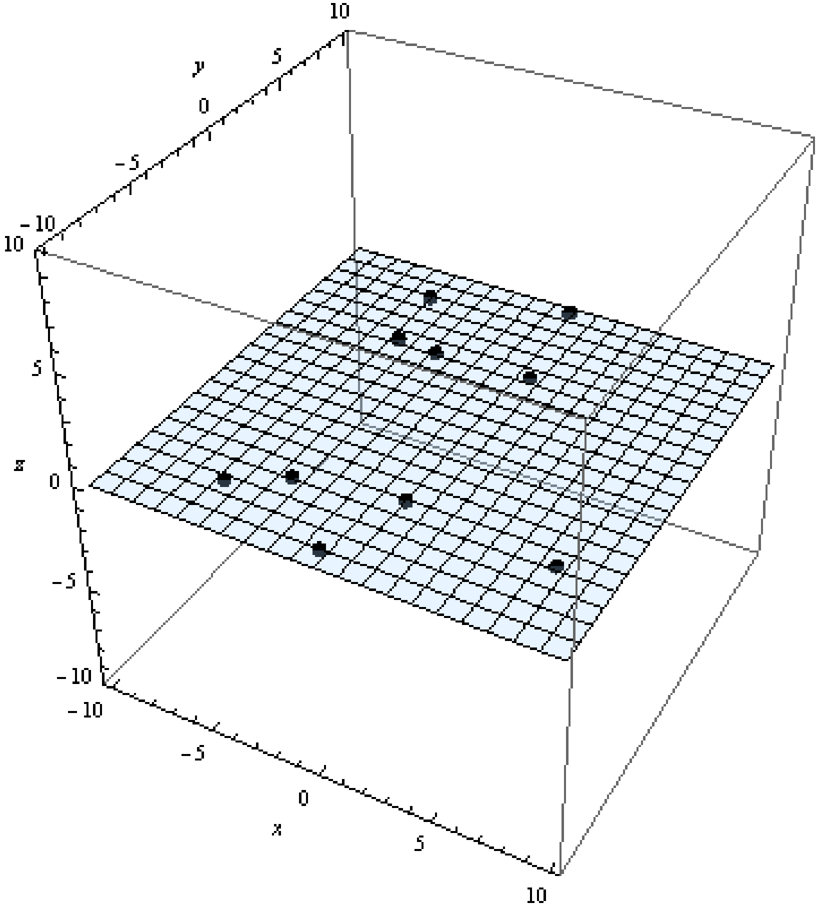
Functions: 30 Degree 2D Rotation

$$f(x, y) = \cos(\theta)x + \sin(\theta)y$$

$$f(x, y) = \cos(30°)x + \sin(30°)y$$

$$f(x, y) = \frac{\sqrt{3}}{2}x + \frac{1}{2}y$$

Functions: 30 Degree 2D Rotation

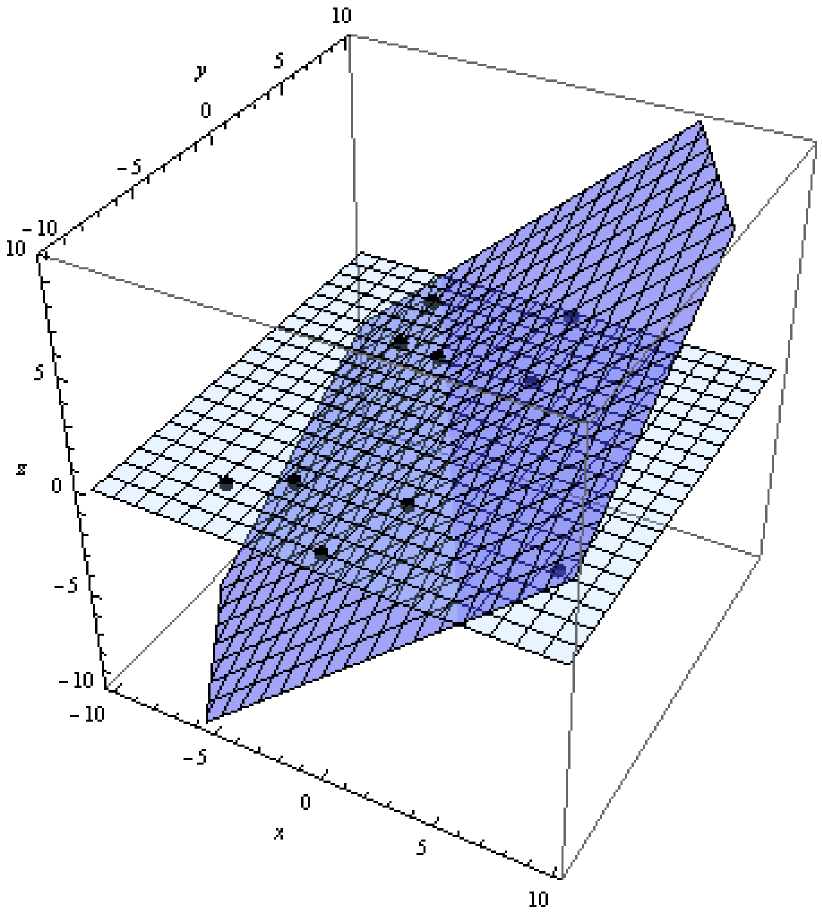


Level Set: $f\left(x, y\right) = c$

where $c \in \mathbb{R}$

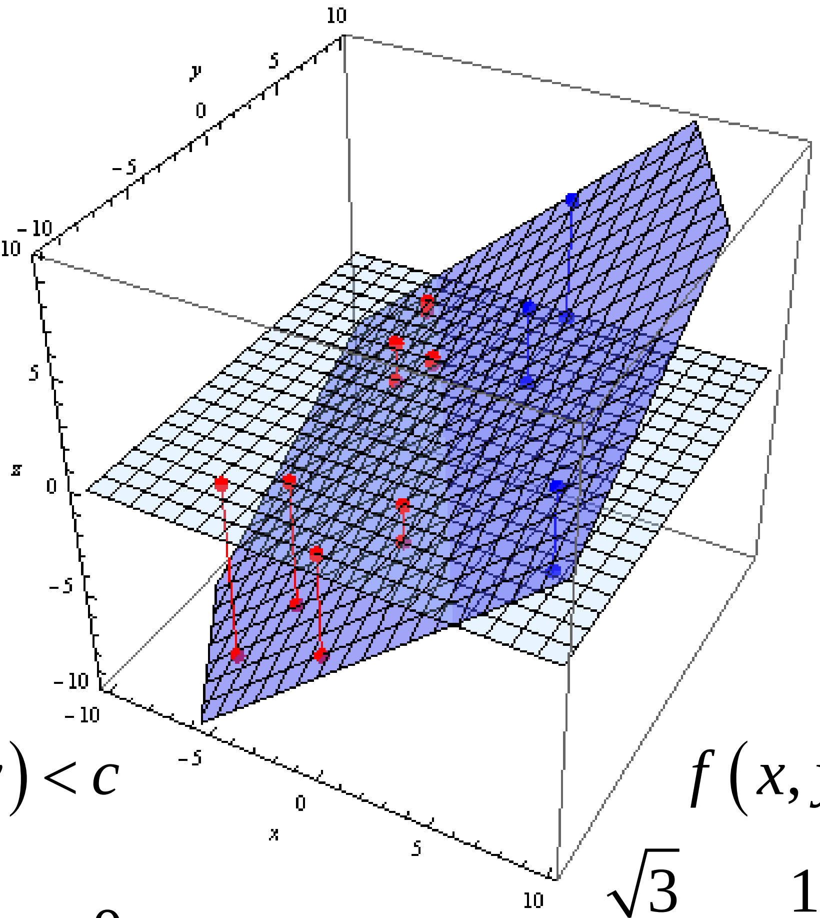
For classification, we can classify points $\left(x, y\right)$ based on the level set such that $f\left(x, y\right) < c$ and $f\left(x, y\right) \geq c$

Functions: 30 Degree 2D Rotation



$$f\left(x, y\right) = \frac{\sqrt{3}}{2} x + \frac{1}{2} y$$

Functions: 30 Degree 2D Rotation



$$f(x,y)=\frac{\sqrt{3}}{2}x+\frac{1}{2}y$$

Level Set:

$$f(x,y)=\quad c$$

Let $c=0$

$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y=0$$

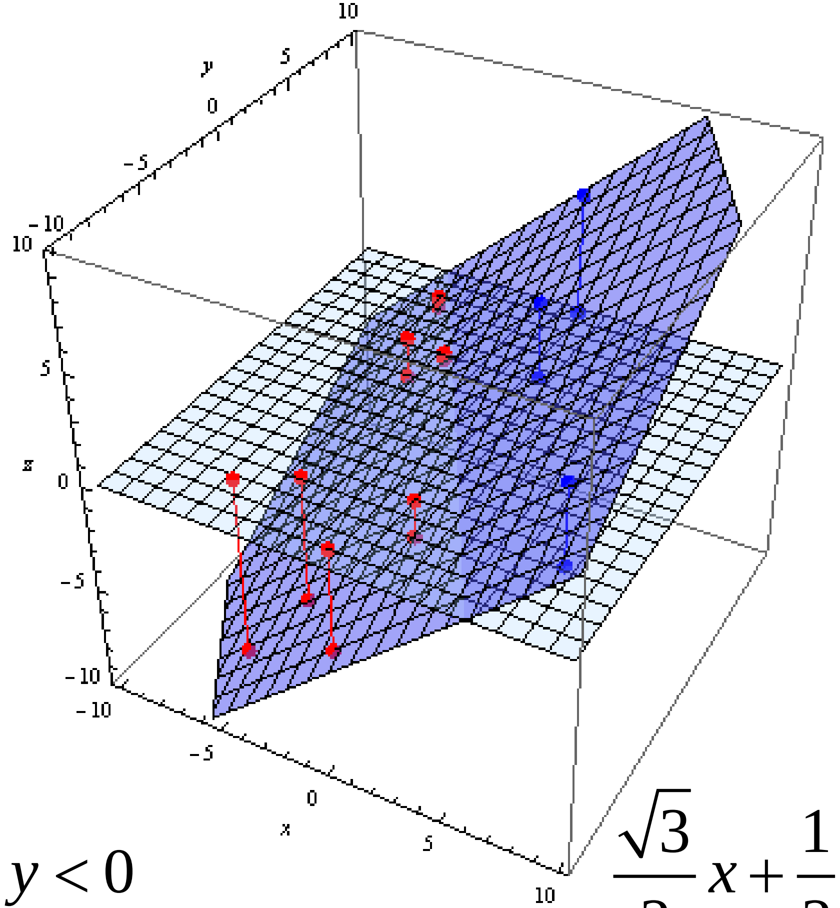
$$f(x,y)<c$$

$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y<0$$

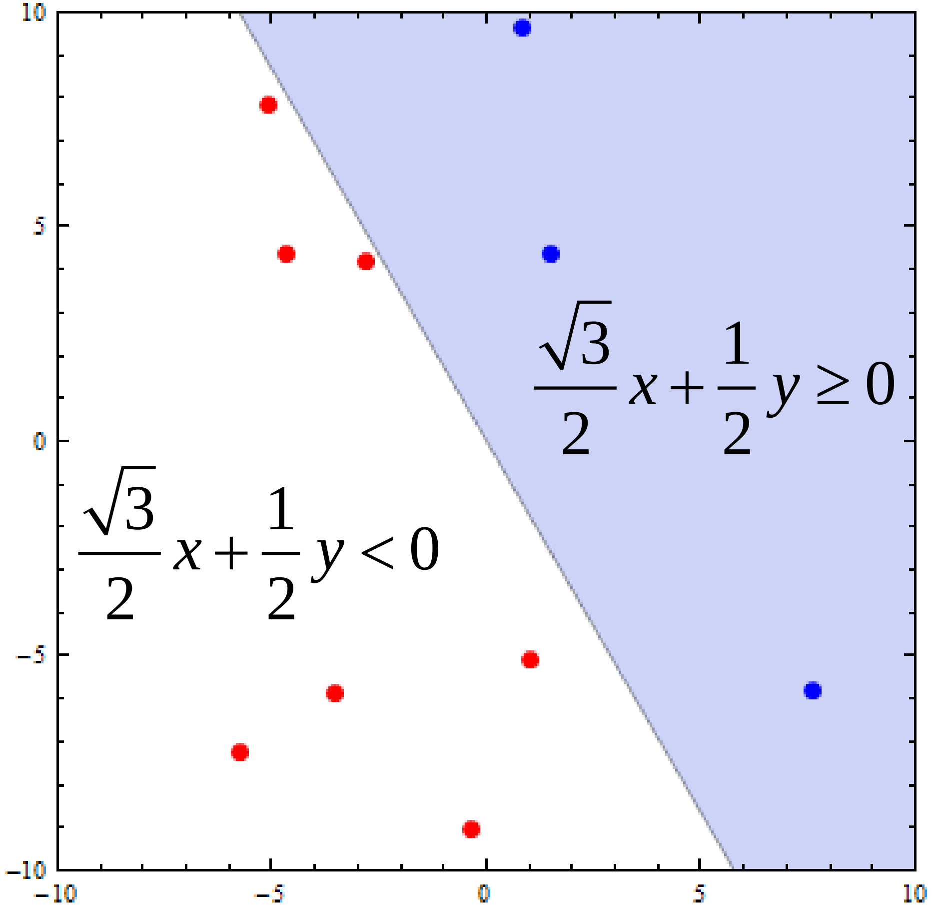
$$f(x,y)\geq c$$

$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y\geq 0$$

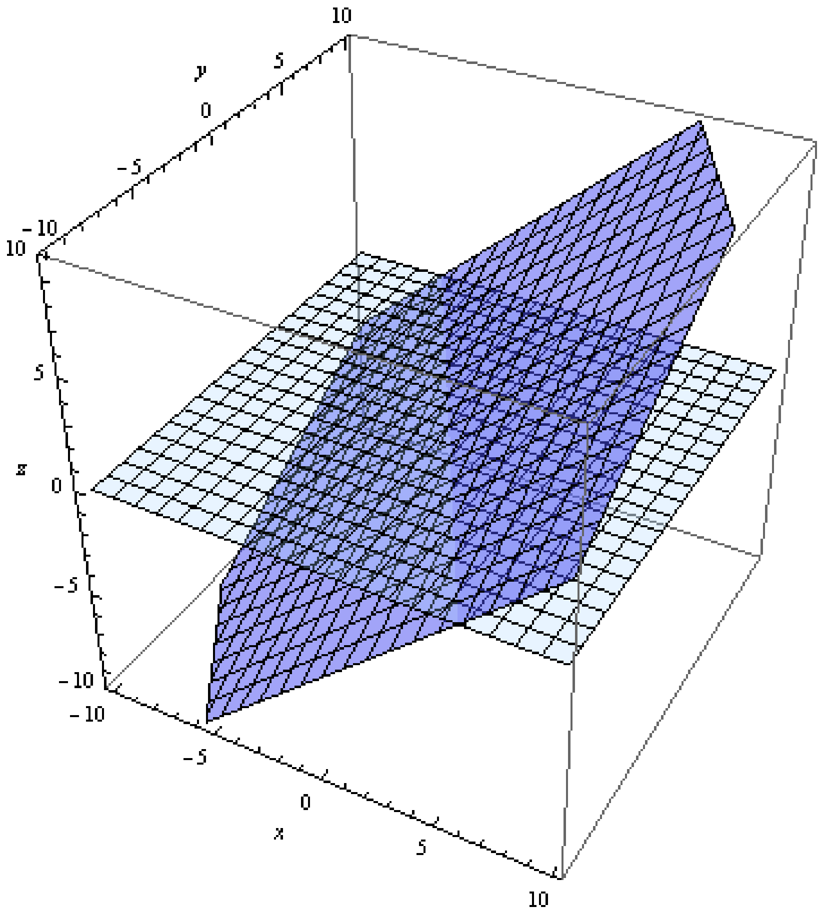
Functions: 30 Degree 2D Rotation/Level Set

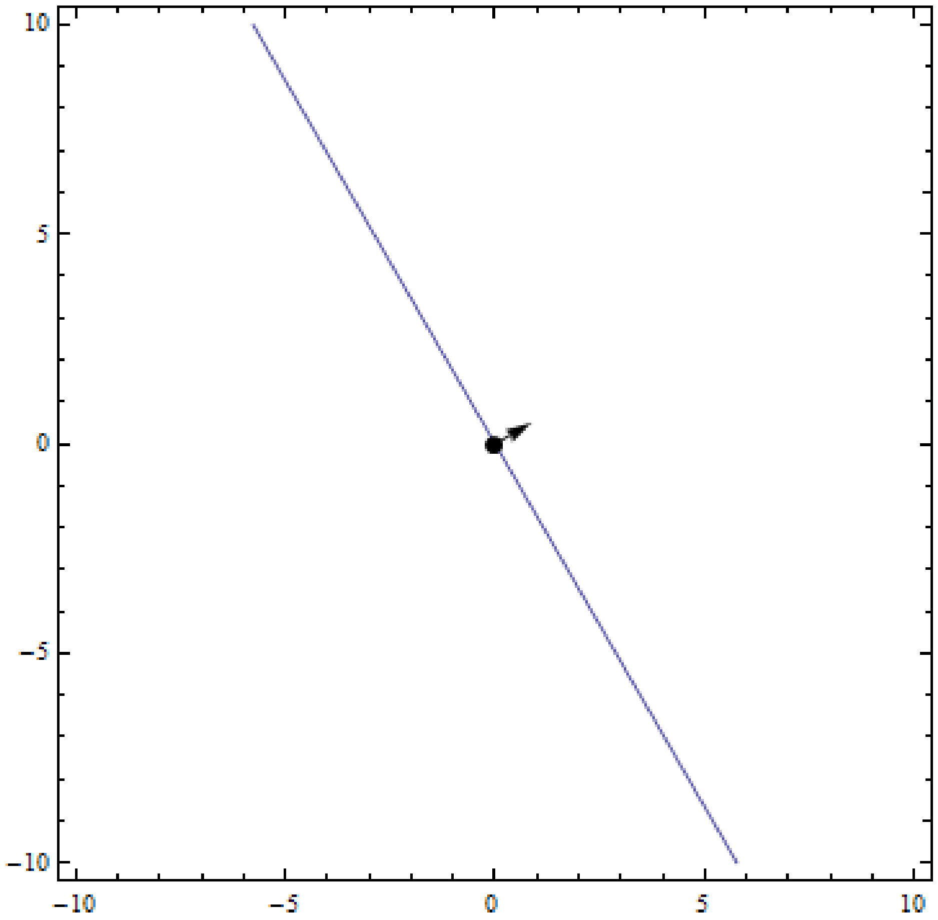


$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y<0 \qquad \frac{\sqrt{3}}{2}x+\frac{1}{2}y\geq 0$$

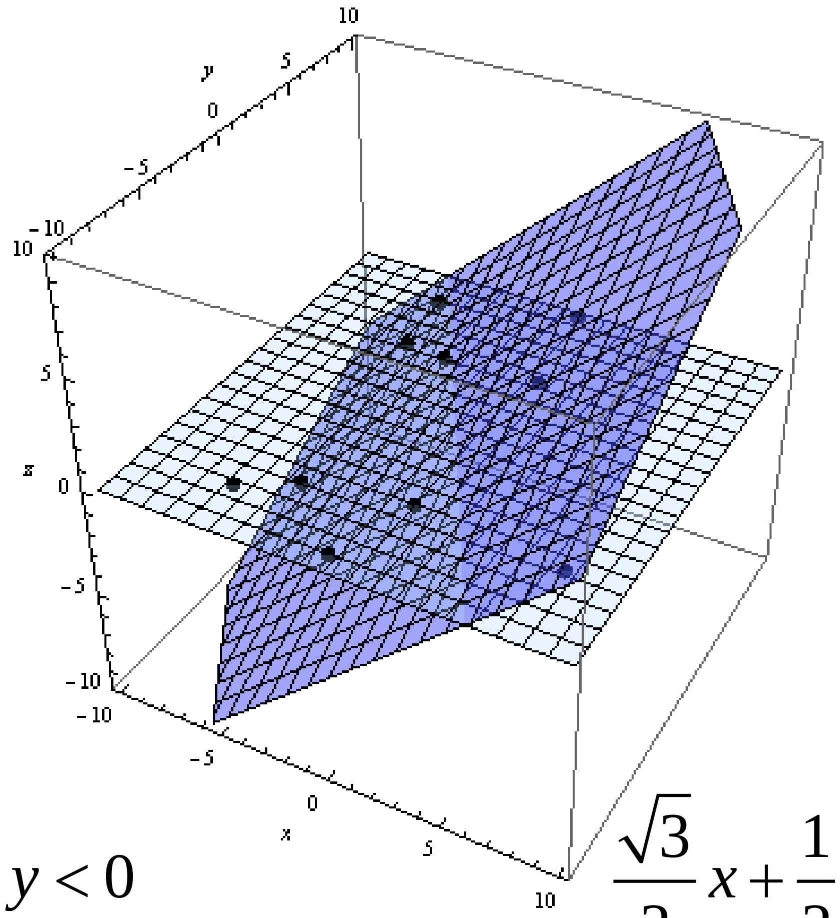


Level Sets / Normal Vector





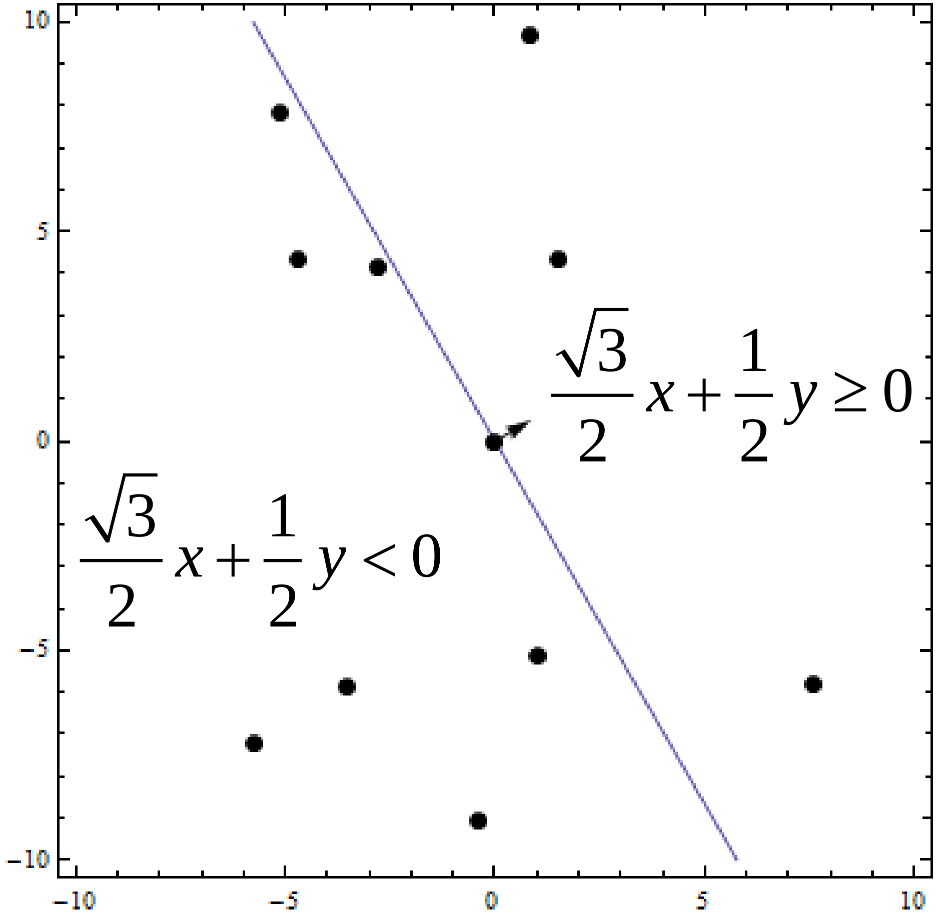
Level Sets / Normal Vector



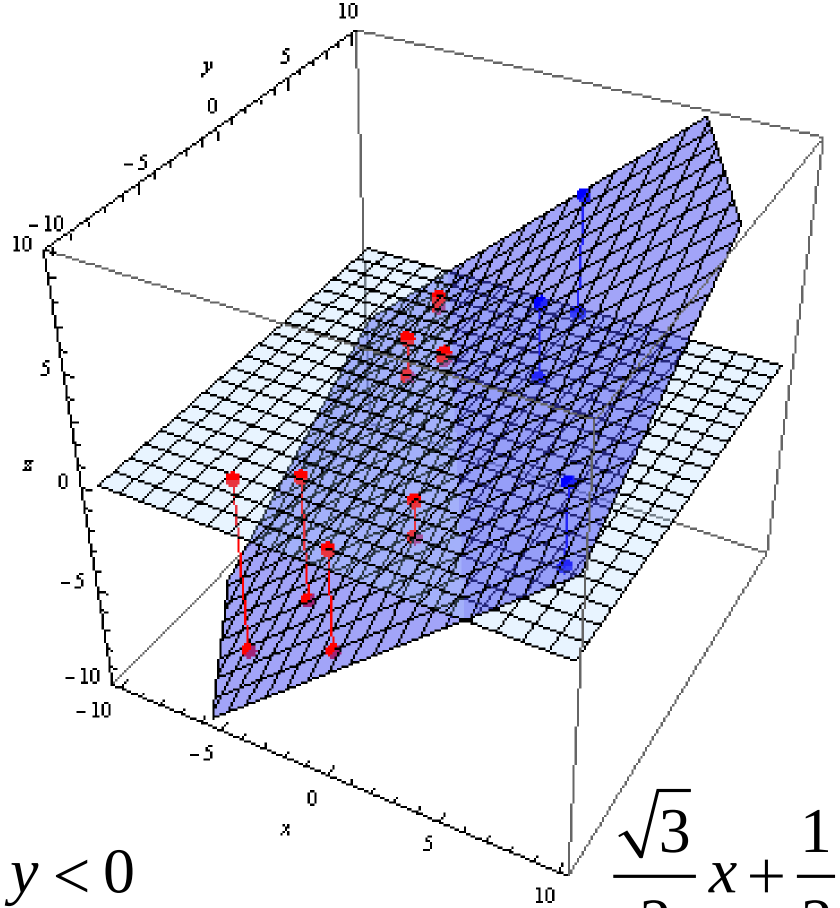
$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y<0$$

$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y\geq 0$$

$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y=0$$

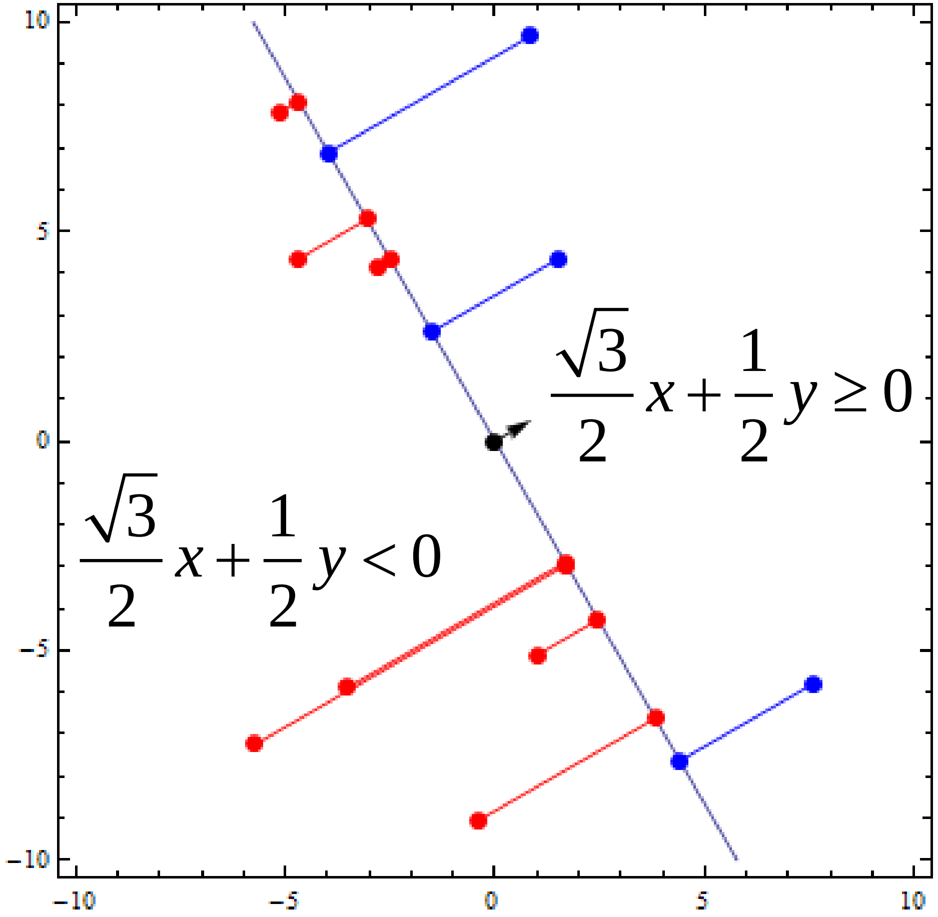


Level Sets / Normal Vector



$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y<0 \qquad \frac{\sqrt{3}}{2}x+\frac{1}{2}y\geq 0$$

$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y=0$$



If Statement: 30 Degree 2D Rotation/Level Set

$$f\left(x,y\right)=\cos\left(30°\right)x+\sin\left(30°\right)y$$

$$f\left(x,y\right)=\frac{\sqrt{3}}{2}x+\frac{1}{2}y$$

$$\text{Level Set: } f\left(x,y\right)=\underset{\text{Let } c=0}{c}$$

$$\frac{\sqrt{3}}{2}x+\frac{1}{2}y=0$$

Classification:

$$f\left(x,y\right)<c\Rightarrow\frac{\sqrt{3}}{2}x+\frac{1}{2}y<0 \qquad f\left(x,y\right)\geq c\Rightarrow\frac{\sqrt{3}}{2}x+\frac{1}{2}y\geq 0$$

If Statement: 30 Degree 2D Rotation/Level Set

```
if ( 0 <= ( sqrt(3)/2 )*x + ( 1/2 )*y ) {
    // Do stuff because the conditional is true
} else {
    // Do stuff because the conditional is false
}
```

If Statement: 30 Degree 2D Rotation/Level Set

```
if ( 0 <= ( sqrt(3)/2 )*f[0] + ( 1/2 )*f[1] ) {
    // Do stuff because the conditional is true
} else {
    // Do stuff because the conditional is false
}
```

If Statement: θ Degree 2D Rotation/Level Set

```
double theta = 30 * (PI/180)
if ( 0 <= cos(theta)*x + sin(theta)*y ) {
    // Do stuff because the conditional is true
} else {
    // Do stuff because the conditional is false
}
```

If Statement: θ Degree 2D Rotation/Level Set

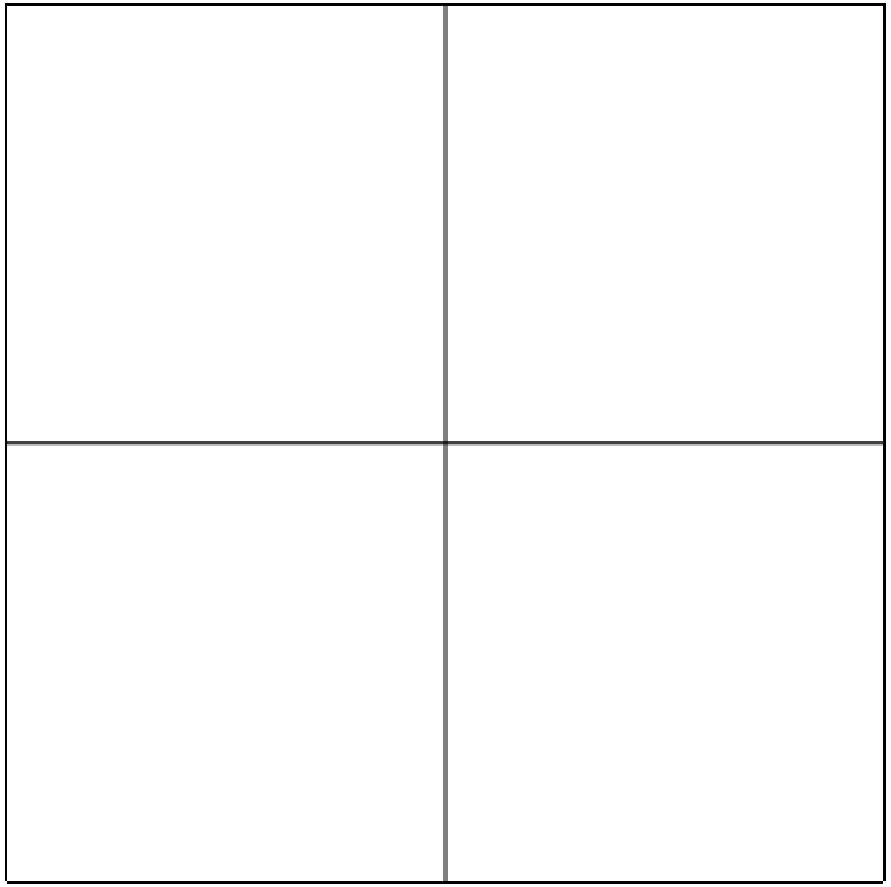
```
if ( 0 <= cos(theta)*f[0] + sin(theta)*f[1] ) {
    // Do stuff because the conditional is true
} else {
    // Do stuff because the conditional is false
}
```

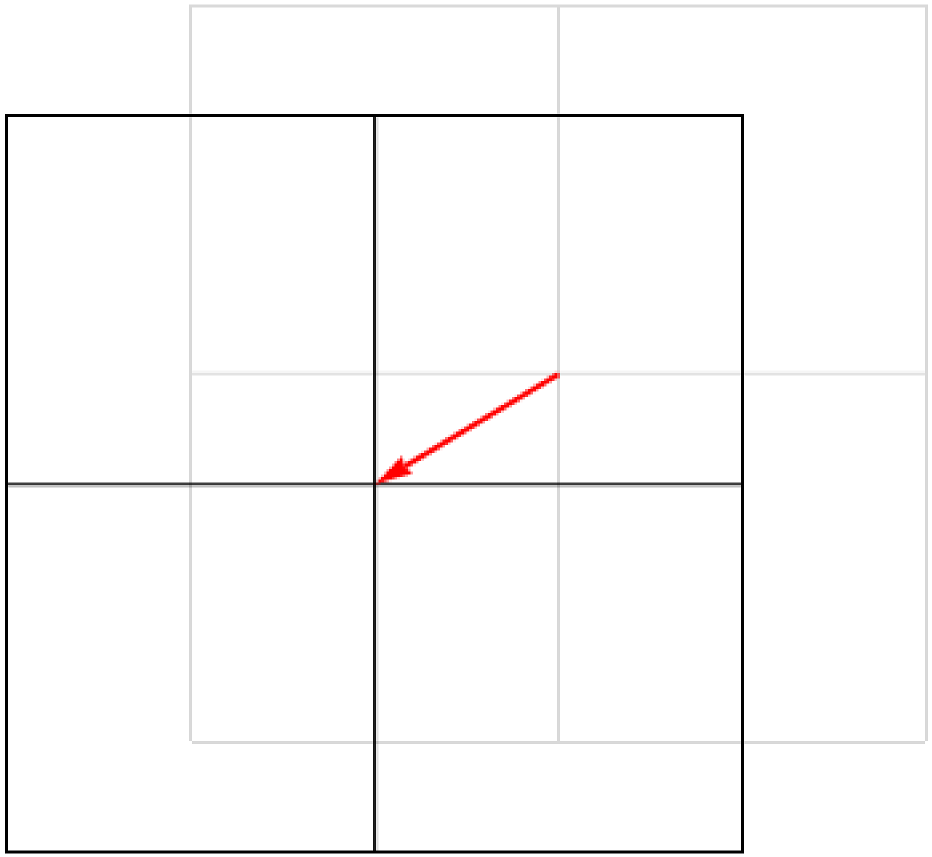
If Statement: θ Degree 2D Rotation/Level Set

```
/*
Filename: main.cpp
To compile and run on linprog4.cs.fsu.edu: g++47 -o main.exe main.cpp -std=c++11 -O3 -Wall -Wextra -Werror -static && ./main.exe
*/
#include <iostream>
#include <cmath>

int main () {
            double const PI{ 3.14159265358979323850 };
            double x{ 7.59389 };
            double y{ -5.7927 };
            double f[]{ x, y };
            double theta{ 30 * ( PI/180 ) };
            if ( 0 <= cos( theta ) * f[ 0 ] + sin( theta ) * f[ 1 ] ) {
                        std::cout << "true -> blue\n";
            } else {
                        std::cout << "false -> red\n";
            }
}
```

Translation





Translation

- 2D Translation Vector:

$$\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Translation

- 2D Translation Vector:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

- Additive Inverse of the 2D Translation Vector:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Translation

- 2D Parametric Translation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Translation

- 2D Function Translation:

$$h(x,y) = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$= (x - x_0, y - y_0)$$

$$h : \mathbb{R}^2 \to \mathbb{R}^2$$

Functions: Composition of Functions

$$f\left(x,y\right)=x$$

$$h\left(x,y\right)=\left(x-x_0,y-y_0\right)$$

$$h:\mathbb{R}^2\rightarrow\mathbb{R}^2 \text{ and } f:\mathbb{R}^2\rightarrow\mathbb{R}^1 \text{ such that } f\circ h:\mathbb{R}^2\rightarrow\mathbb{R}^1$$

$$f\left(h\left(x,y\right)\right)=\left(x-x_0\right)$$

$$f\left(x,y\right)=x-x_0$$

Functions: Composition of Functions

$$f(x,y)=\frac{\sqrt{3}}{2}x+\frac{1}{2}y$$

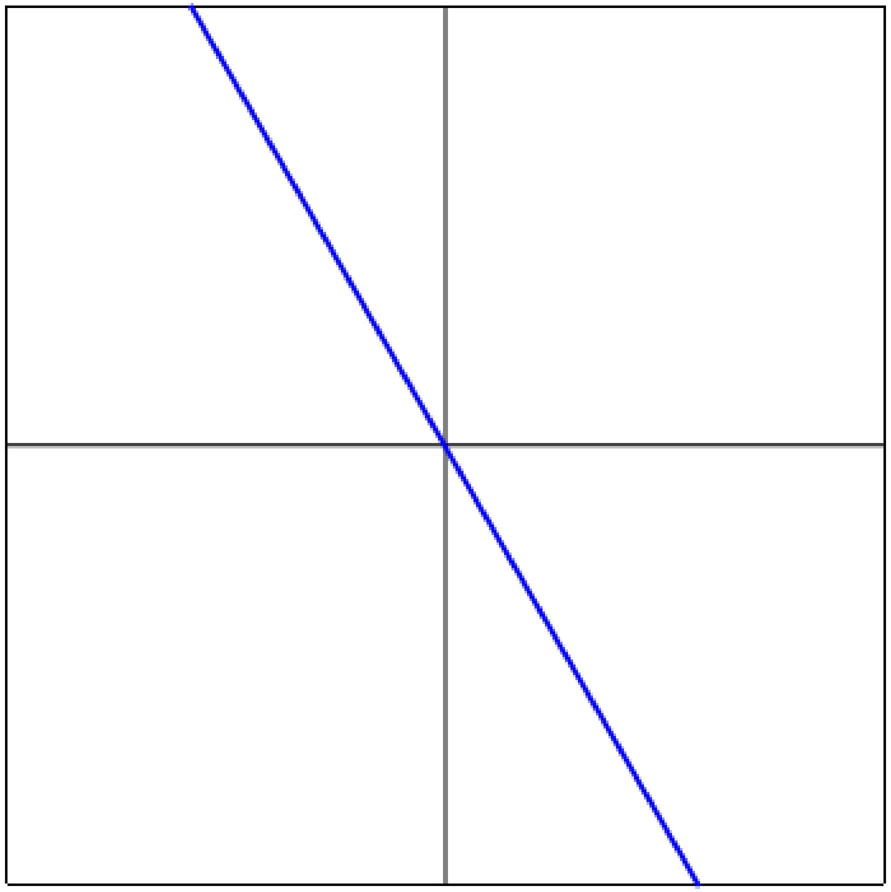
$$h(x,y)=(x-x_0,y-y_0)$$

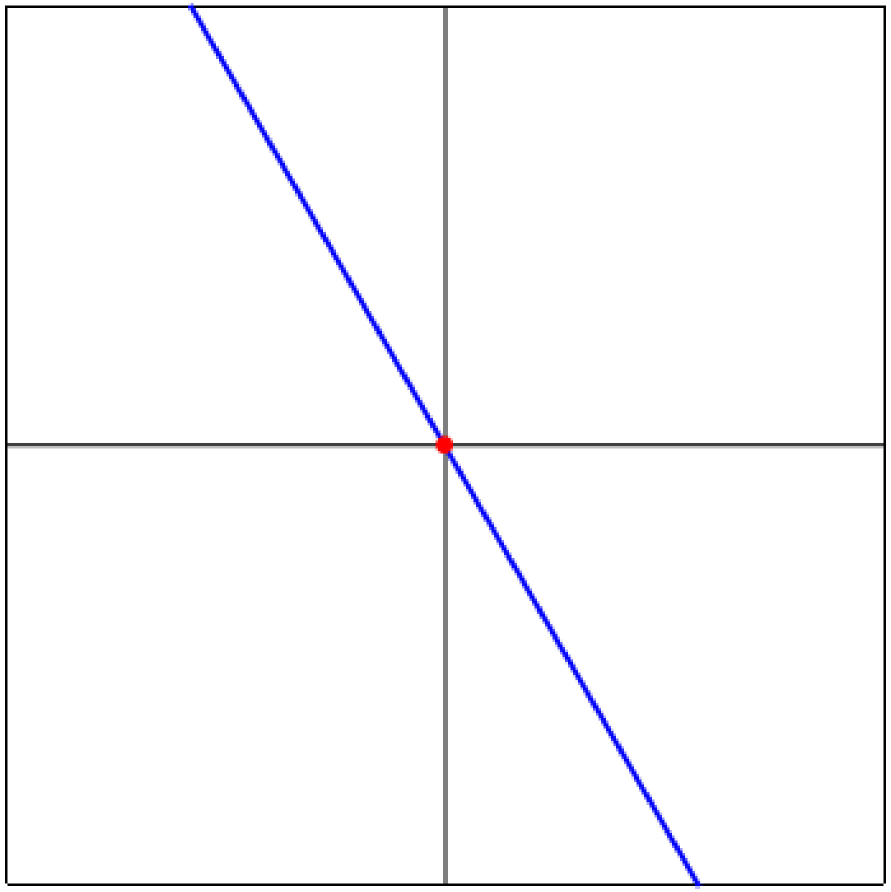
$$h:\mathbb{R}^2\to\mathbb{R}^2 \text{ and } f:\mathbb{R}^2\to\mathbb{R}^1 \text{ such that } f\circ h:\mathbb{R}^2\to\mathbb{R}^1$$

$$f(h(x,y))=\frac{\sqrt{3}}{2}(x-x_0)+\frac{1}{2}(y-y_0)$$

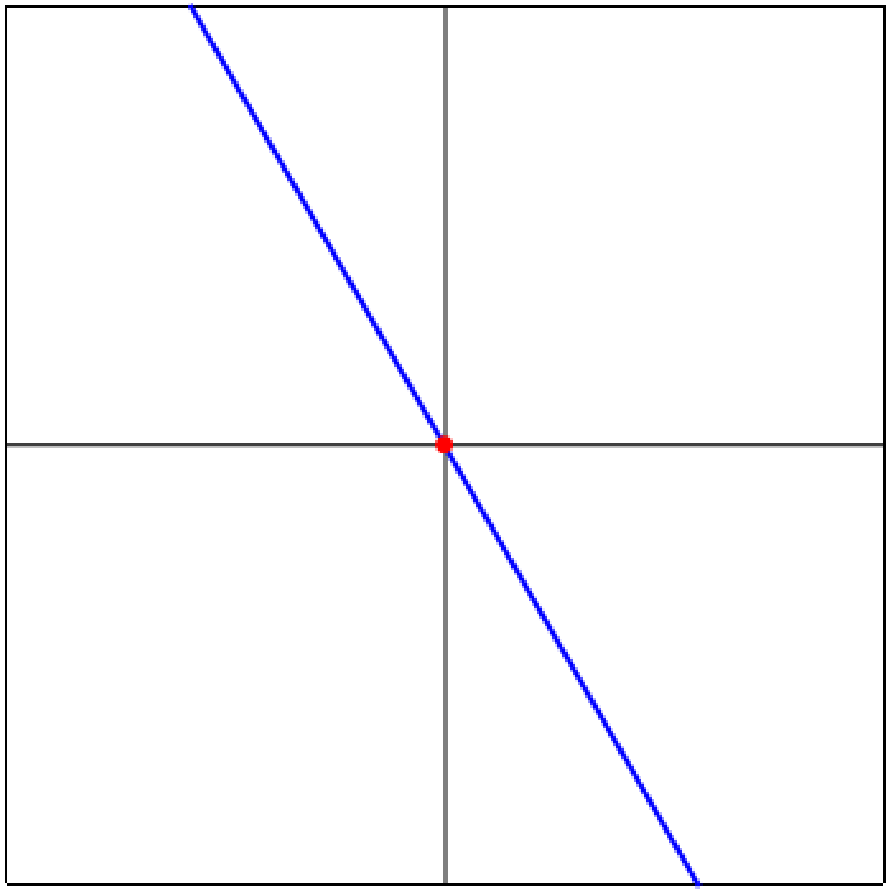
$$f(x,y)=\frac{\sqrt{3}}{2}(x-x_0)+\frac{1}{2}(y-y_0)$$

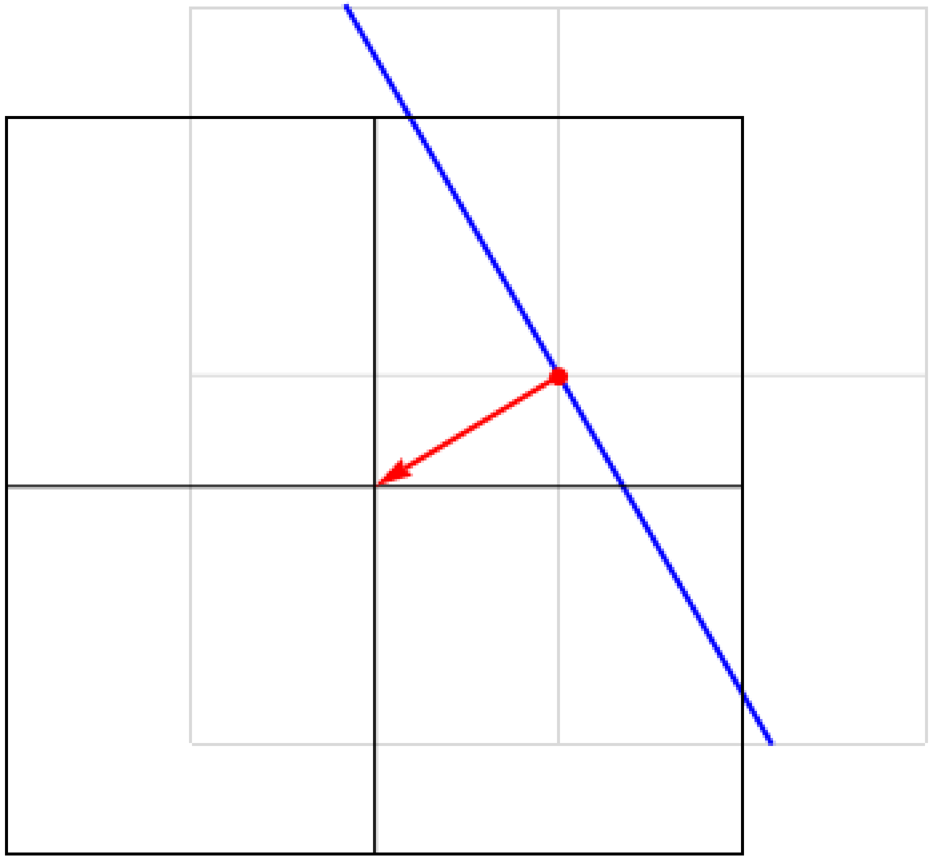
Functions:
(5,3) 2D Translation of 30 Degree 2D Rotation



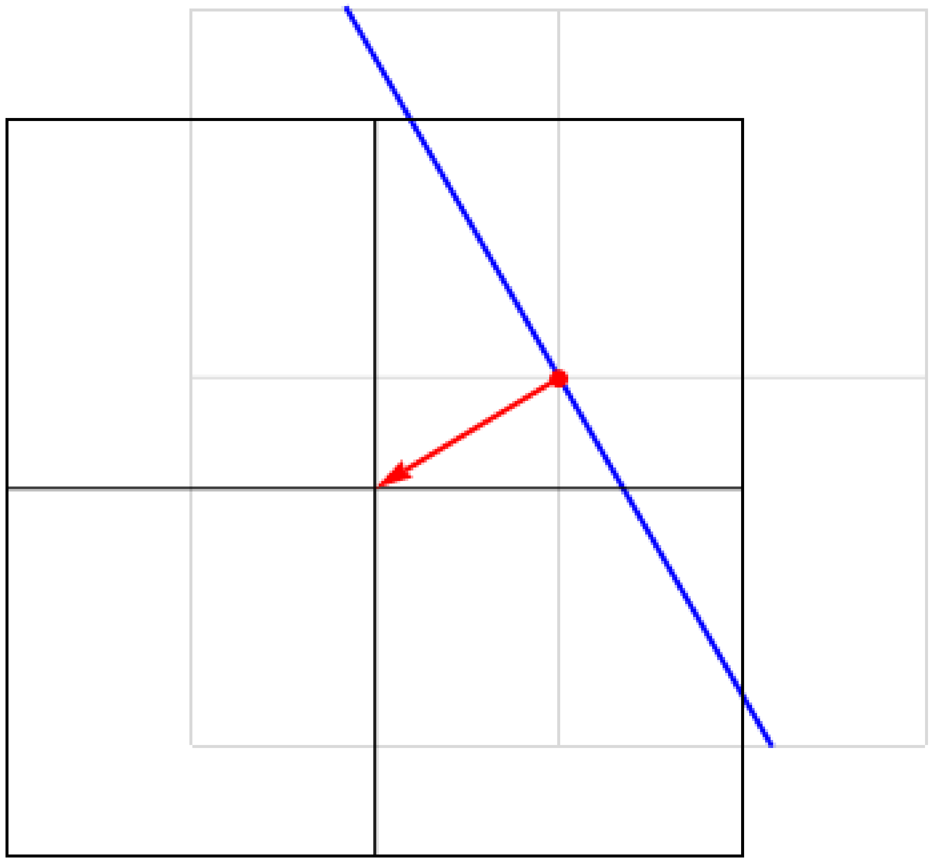


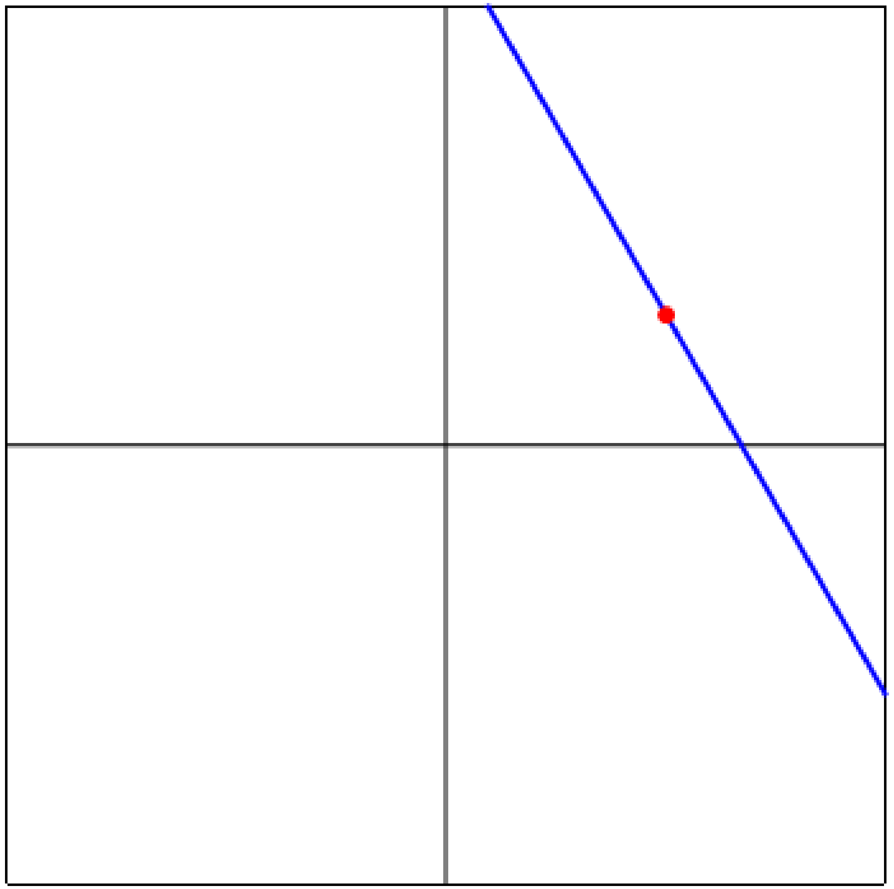
Functions:
(5,3) 2D Translation of 30 Degree 2D Rotation





Functions:
(5,3) 2D Translation of 30 Degree 2D Rotation





Functions:
(5,3) 2D Translation of 30 Degree 2D Rotation

$$f(x,y)=\frac{\sqrt{3}}{2}(x-x_0)+\frac{1}{2}(y-y_0)$$

For a translation to $(5,3)$:

$$f(x,y)=\frac{\sqrt{3}}{2}(x-(5))+\frac{1}{2}(y-(3))$$

$$f(x,y)=\frac{\sqrt{3}}{2}(x-5)+\frac{1}{2}(y-3)$$

Conclusions

- Either Level Sets or Normal Vectors can be used for linear classification.
- Both Levels Sets and Normal Vectors convert information into Real Numbers in order to classify that data based on negative, zero, or positive values.
- Rotations and Translations are some of the types of linear transformations that allow for Level Sets and Normal Vectors to be modified to change the way data is classified.
- Order of applying Rotations and Translations matters. If you want to rotate a function around its translation point, apply the Rotation first followed by the Translation. If you want to rotate a function around the origin, apply the Translation first followed by the Rotation.
- Levels Sets and Normal Vectors can be used for Binary Space Partitioning.